Theory of Computation
The Class NP

Search vs. Verification

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Which tasks are are easier?

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- Doing a homework assignment
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- Finding 1000 Facebook users who are all friends

- Reviewing a movie
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- Checking that a proof is valid
- Checking if 1000 Facebook users are all friends

3-CNF Formulas

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B) $F = (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_5)$
C) $F = (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_3 \vee \neg x_4)$
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The language $\mathrm{3\text{-}SAT}$

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Which of the following formulas are in $\mathrm{3\text{-}SAT}$?

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- $O(2^n) \cdot \text{poly-time} \in \text{EXP}$

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- Can 3-SAT be solved in polynomial time?
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- But we also have reason to believe that 3-SAT is easier than some other problems in EXP
- 3-SAT can be **verified** in polynomial time

3-SAT search vs. verification

Is the following 3-CNF formula satisfiable?

$$
\begin{aligned}
F = & (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \\
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The string c is sometimes called a **certificate**, **witness**, or **proof** that $w \in L$

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- $O(m \cdot n) =$ poly-time verification

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- Nondeterminism doesn't make our machines more *robust*
- **Does nondeterminism make our machines faster?**

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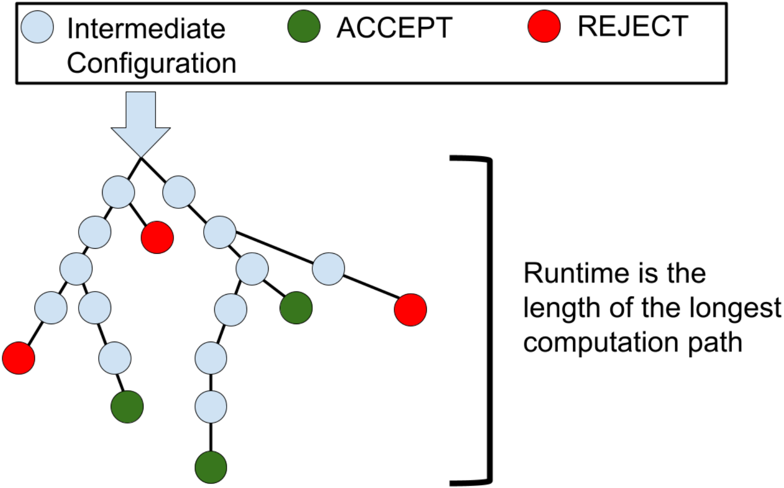
  - It only takes a polynomial amount of time to "guess" the solution

Nondeterministic Runtimes



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$$\mathrm{NP} = \bigcup_{c} \mathrm{NTIME}(T(n^c))$$

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- If F is not satisfiable, every computation path will reject

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NP and verification

Let's re-examine the nondeterministic 3-SAT algorithm

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NP and verification

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Input: ~~A formula F with n variables and m clauses~~
A string w

1. ~~*Nondeterministically guess* truth assignment A~~
   Nondeterministically guess a certificate c
2. ~~Check if A satisfies the formula F~~
   Check if c proves that $w \in L$
3. ~~Accept F if A satisfies F. Reject otherwise~~
   Accept if c proves that $w \in L$. Reject otherwise

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- **Nondeterministic machines are efficient when there is a short, easily verified certificate**

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- Suppose $L \in \mathrm{NP}$. Then L is recognized by an NTM M that runs in polynomial time
- Construct a verifier V that takes a string w and an *accepting computation history* H as input
- Because M runs in poly-time, the length of computation history is polynomially bounded
- We can verify that H is a computation history for which M accepts w in poly-time

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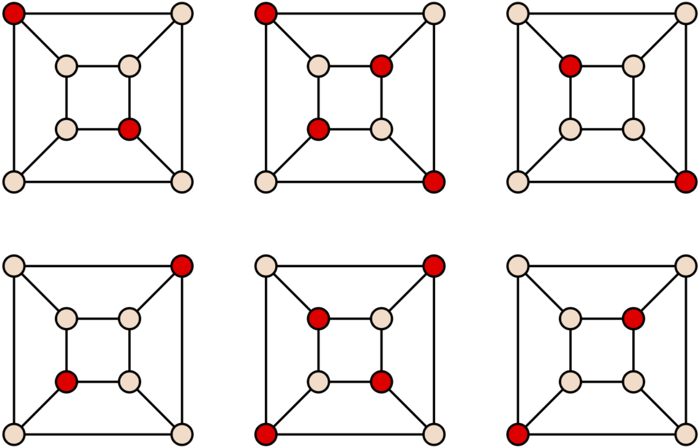

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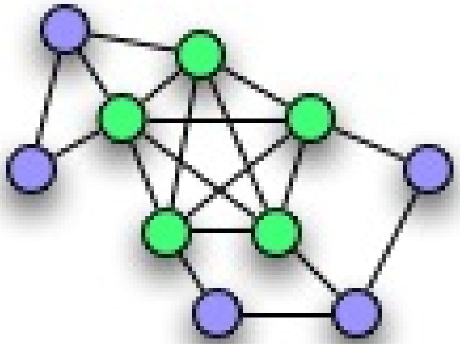

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Solution: It is impossible; $6 + 8 + 10 = 24 < 101$

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Activity: Search vs. Verification