Theory of Computation
Complexity classes, P, EXP

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  - **What problems can't be solved efficiently?**

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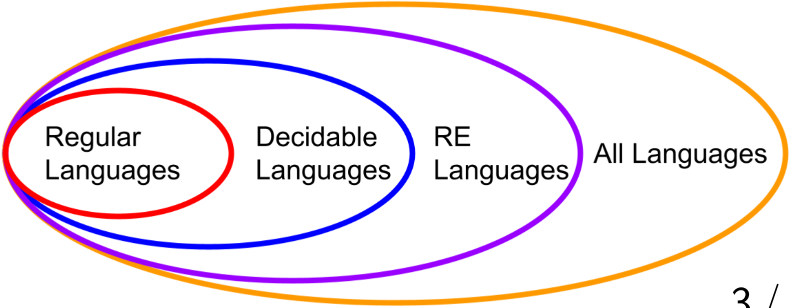

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  - In fact, $L \in \mathrm{TIME}(n \log(n))$ - see Sipser

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- In this course, we will use P as a proxy for "tractable" problems

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- An 8-byte unary integer cannot represent numbers bigger than 32!!!
- If the input is in binary (or base 10 or base 16), we have to be careful about runtime analysis

Runtime with numeric inputs

What is the running time of this algorithm?

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- This is exponential in the length of the input!!!

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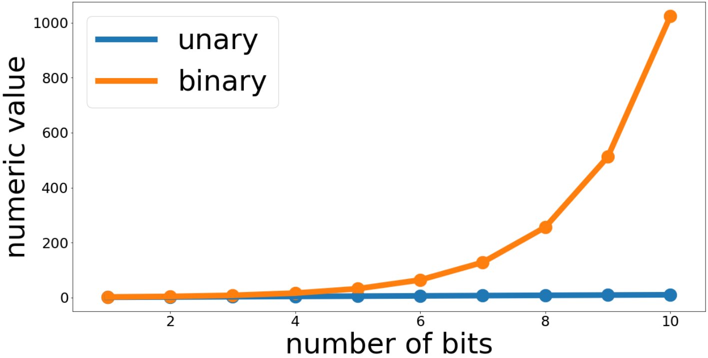

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- ...which is $O(2^n)$ in the *length* of $\langle x, y \rangle$

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78 ÷ 66 = 1 remainder 12 $\quad (78 = 66 \times 1 + 12)$

66 ÷ 12 = 5 remainder 6 $\quad (66 = 12 \times 5 + 6)$

12 ÷ 6 = 2 remainder 0 $\quad (12 = 6 \times 2 + 0)$

6 = Greatest Common Factor

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- **Case 1:** $y \leq \frac{x}{2}$. Then $x \,\%\, y < y \leq \frac{x}{2}$
- **Case 2:** $y > \frac{x}{2}$. Then $x \,\%\, y = x - y < \frac{x}{2}$

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- The number of times we can cut the input in half is $\log(\max\{x, y\}) = O(|\langle x, y \rangle|)$

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$$\text{UNARY-SUBSET-SUM} = \left\{ \langle B | x_1, x_2, \ldots x_n \rangle \middle| \begin{array}{c} \text{B is unary} \\ \text{there is a combination of } x_i \text{ (no repeats)} \\ \text{that add up to B} \end{array} \right\}$$

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Example: $\langle 101 | 6, 8, 10 \rangle$
Solution: It is impossible; $6 + 8 + 10 = 24 < 101$

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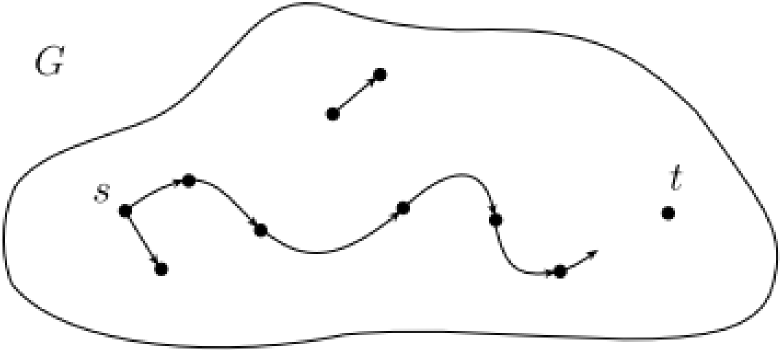

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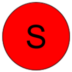
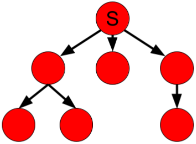
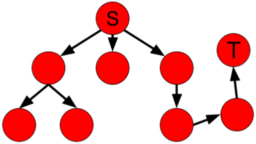


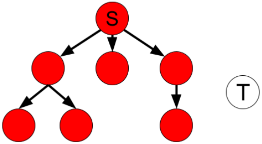
2. Mark all neighbors of S (and their neighbors, and so on)



3. Continue until T gets marked…



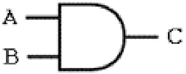
4. ...or until we can't mark further



Logical symbols

Logical symbols

AND

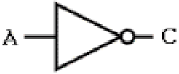


Inputs		Output
A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

OR

Inputs		Output
A	B	C
0	0	0
0	1	1
1	0	1
1	1	1

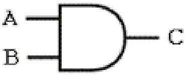
NOT



Input	Output
A	C
0	1
1	0

Logical symbols

AND



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A	B	C
0	0	0
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OR

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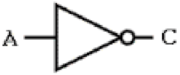
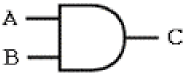


Input	Output
A	C
0	1
1	0

- AND ($\wedge$): all inputs must be TRUE

Logical symbols

AND



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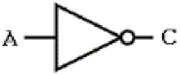

Input	Output
A	C
0	1
1	0

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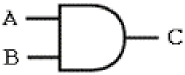

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A	B	C
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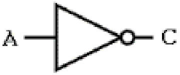

Input	Output
A	C
0	1
1	0

- AND ($\wedge$): all inputs must be TRUE
- OR ($\vee$): at least one input must be TRUE
- NOT ($\neg$): input must be FALSE

Logical symbol practice

Suppose $x = \mathrm{TRUE}, y = \mathrm{TRUE}, z = \mathrm{FALSE}.$
Which of the following expressions are TRUE?

A) x

B) z

C) $y \vee z$

D) $\neg(x \wedge y)$

E) $(x \vee y) \wedge (y \vee z)$

F) $\neg x \vee (\neg y \vee \neg z)$

G) $(x \wedge y) \wedge (y \wedge z)$

H) $(x \vee y) \wedge (z \vee z \vee z)$

Logical symbol practice

Suppose $x = \text{TRUE}, y = \text{TRUE}, z = \text{FALSE}$.
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C) $y \vee z$ ✓

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Conjunctive Normal Form

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2. Each clause is conjunction of several **variables**

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Examples:

- $(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5)$

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Examples:

- $(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5)$
- $(x_1 \vee \neg x_1) \wedge (x_2 \vee x_3 \vee x_4 \vee x_5 \vee \neg x_1) \wedge (\neg x_2)$

Conjunctive Normal Form

Which of the following expressions are in conjunctive normal form?

A) (x_1)
B) (x_2)
C) $(\neg x_1 \vee \neg x_1)$
D) $\neg(x_1 \vee x_1)$
E) $(x_1 \wedge x_2 \wedge x_3) \vee (x_4 \wedge x_5)$
F) $(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$
G) $(x_1 \vee x_2 \vee x_3) \vee (\neg x_1 \vee \neg x_2)$
H) $(x_1 \wedge x_2 \wedge x_3) \wedge (\neg x_1 \wedge \neg x_2)$

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C) $(\neg x_1 \vee \neg x_1)$ ✓
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E) $(x_1 \wedge x_2 \wedge x_3) \vee (x_4 \wedge x_5)$
F) $(x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ ✓
G) $(x_1 \vee x_2 \vee x_3) \vee (\neg x_1 \vee \neg x_2)$
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  - **Note:** If x_i is FALSE then $\neg x_i$ is TRUE
- A CNF clause is **satisfied** if at least one of its variables is TRUE
- A CNF formula is satisfied if *all* of its clauses are satisfied
- A CNF formula is **satisfiable** if there *exists* a satisfying assignment

CNF Satisfying Assignment

$$F = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee x_3 \vee x_4) \wedge (x_2) \wedge (\neg x_5 \vee \neg x_1)$$

$x_1 = x_4 = x_5 = \text{TRUE}$
$x_2 = x_3 = \text{FALSE}$

Which clauses are satisfied?

A) $(x_1 \vee x_2 \vee x_3)$
B) $(\neg x_1 \vee x_3 \vee x_4)$
C) (x_2)
D) $(\neg x_5 \vee \neg x_1)$

CNF Satisfying Assignment

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Which clauses are satisfied?

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CNF Satisfiability

$x_1 = x_4 = x_5 = \mathrm{TRUE}$
$x_2 = x_3 = \mathrm{FALSE}$

Which of the following formulas are satisfied?

A) $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_4 \vee x_5)$

B) $F = (x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4) \wedge (x_5)$

C) $F = (x_1) \wedge (x_2) \wedge (x_3) \wedge (x_4) \wedge (x_5)$

D) $F = (\neg x_1 \vee \neg x_4 \vee x_5) \wedge (x_2 \vee x_3)$

CNF Satisfiability

$x_1 = x_4 = x_5 = \text{TRUE}$
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Which of the following formulas are satisfied?

A) $F = (x_1 \vee x_2 \vee \neg x_3) \wedge (x_4 \vee x_5)$ ✓

B) $F = (x_1 \vee \neg x_2 \vee x_3 \vee \neg x_4) \wedge (x_5)$ ✓

C) $F = (x_1) \wedge (x_2) \wedge (x_3) \wedge (x_4) \wedge (x_5)$

D) $F = (\neg x_1 \vee \neg x_4 \vee x_5) \wedge (x_2 \vee x_3)$

CNF Satisfying Assignment

Which of the following formulas are satisfiable?

A) $F = (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$

B) $F = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$

C) $F = (x_1) \wedge (\neg x_2)$

D) $F = (x_1) \wedge (\neg x_1)$

CNF Satisfying Assignment

Which of the following formulas are satisfiable?

A) $F = (x_1 \vee x_2 \vee x_3) \wedge (x_4 \vee x_5 \vee x_6)$ ✓

B) $F = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_3)$ ✓

C) $F = (x_1) \wedge (\neg x_2)$ ✓

D) $F = (x_1) \wedge (\neg x_1)$

CNF Satisiability

Is the following formula satisfiable?

$(x_1 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_1 \vee \neg x_3)$

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Is the following formula satisfiable?

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These four clauses can't all be satisfied!

The language 2-SAT

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$$\text{2-SAT} = \{F \mid F \text{ is a satisfiable 2-CNF formula}\}$$

Which of these formulas are in the language 2-SAT?

A) $(x_1 \vee x_2) \wedge (x_3 \vee x_4)$
B) $(x_1 \vee x_1) \wedge (\neg x_1 \vee \neg x_1)$
C) $(x_1) \wedge (x_2) \wedge (x_3)$
D) $(x_1 \vee x_2 \vee x_3)$

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Def: A **2-CNF Formula** is a CNF formula with at most 2 variables in each clause

$$\text{2-SAT} = \{F | F \text{ is a satisfiable 2-CNF formula}\}$$

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C) $(x_1) \wedge (x_2) \wedge (x_3)$ ✓
D) $(x_1 \vee x_2 \vee x_3)$

Satisfying a 2-CNF Formula

Consider the following formula:

$$F = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_4 \vee x_1)$$

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- If x_1 is FALSE then x_2 must be FALSE

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- If x_2 is FALSE, then x_3 must be TRUE

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- If x_3 is TRUE then x_4 must be FALSE

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- If x_3 is TRUE then x_4 must be FALSE
- If x_4 is FALSE then x_1 must be TRUE

Satisfy

Consi

$F = ($ … $\lor x_1)$

Satisfying a 2-CNF Formula

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$$F = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (x_4 \vee \neg x_1)$$

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- If x_4 is TRUE, then x_3 must be FALSE
- If x_3 is FALSE then x_2 must be TRUE

Satisfying a 2-CNF Formula

Consider the following formula:

$$F = (x_1 \lor \neg x_2) \land (x_2 \lor x_3) \land (\neg x_3 \lor \neg x_4) \land (x_4 \lor \neg x_1)$$

- If x_1 is TRUE then x_4 must be TRUE
- If x_4 is TRUE, then x_3 must be FALSE
- If x_3 is FALSE then x_2 must be TRUE
- If x_2 is TRUE then x_1 must be TRUE - which it is!

2-SAT implication graph

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- Suppose we have a clause $C = (x_i \vee x_j)$

2-SAT implication graph

- Suppose we have a clause $C = (x_i \lor x_j)$
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  - $\neg x_j \implies x_i$
- If $x_i \implies x_j$ and $x_j \implies x_k$ then $x_i \implies x_k$ (transitive property)

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  - Every edge is an implication

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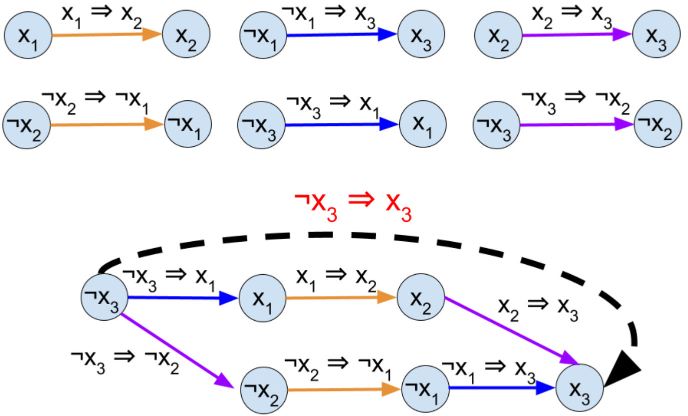
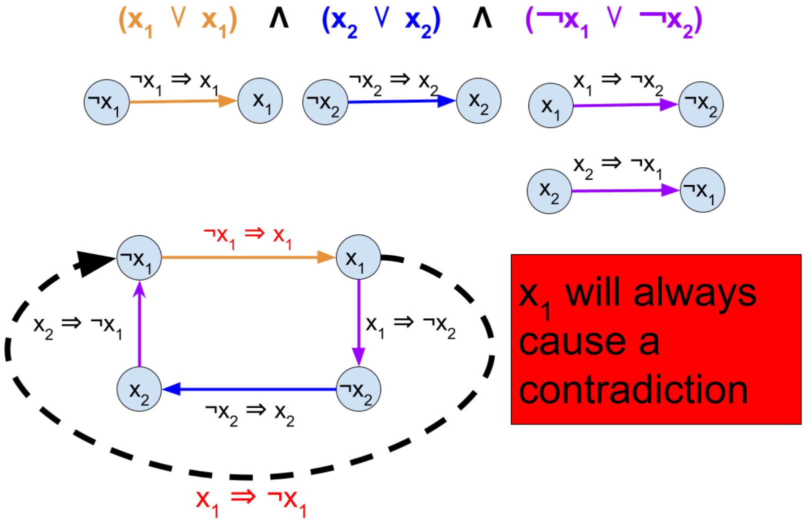
- We can use an **implication graph** to represent these relationships
  - Every node is variable
  - Every edge is an implication
  - **Every path is a (transitive) implication**

2-SAT implication graph

$(\neg x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (\neg x_2 \vee x_3)$



2-SAT implication graph



2-SAT $\in$ P

2-SAT ∈ P

Input: a formula F with n variables and m clauses

2-SAT $\in$ P

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1. Create the implication graph for F

2-SAT $\in$ P

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2. For every variable x_i do the following

2-SAT $\in$ P

Input: a formula F with n variables and m clauses

1. Create the implication graph for F
2. For every variable x_i do the following
   2.1 Check if there is a path from x_i to $\neg x_i$

2-SAT $\in$ P

Input: a formula F with n variables and m clauses

1. Create the implication graph for F
2. For every variable x_i do the following
   2.1 Check if there is a path from x_i to $\neg x_i$
   2.2 Check if there is a path from $\neg x_i$ to x_i

2-SAT $\in$ P

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- $O(n) + O(m) + O(n) \cdot$ poly-time $\in \mathrm{P}$

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- EXP languages are considered "intractable"

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  - Can every exponential-time algorithm be converted to a polynomial-time algorithm?

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$$\mathrm{P} \subseteq \mathrm{TIME}(2^n) \subsetneq \mathrm{TIME}((2^{2n})^3) \subseteq \mathrm{EXP}$$