DFA Closure Properties

Arjun Chandrasekhar

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- Let's design DFAs to recognize L and L^c

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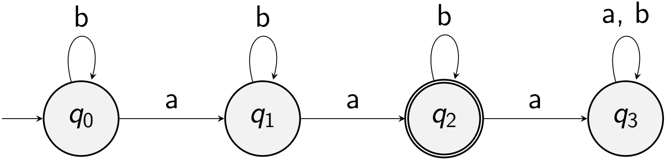

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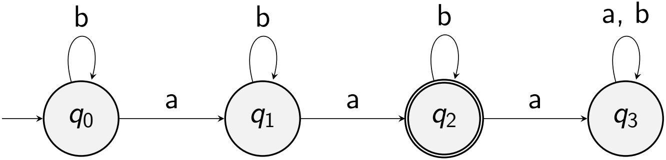

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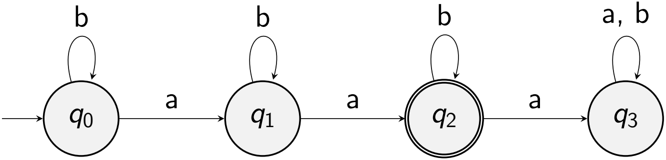

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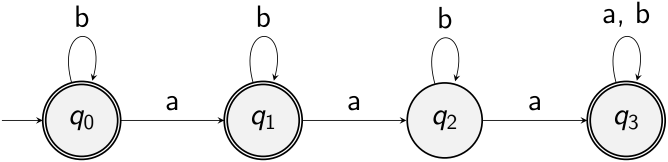

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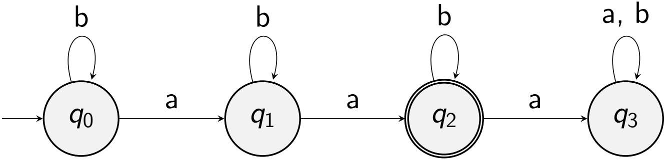

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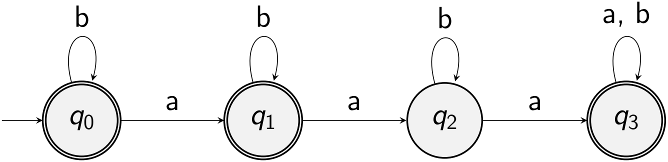
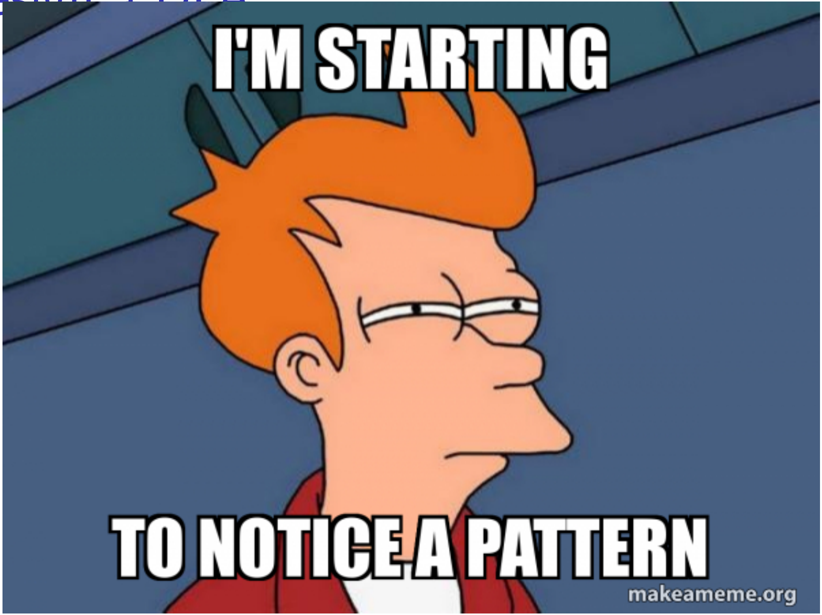


Notice a pattern?

Design a DFA



Closure of regular languages under complement

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Proposition: Regular languages are closed under complement

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- What do we know about L?
  - L is regular...
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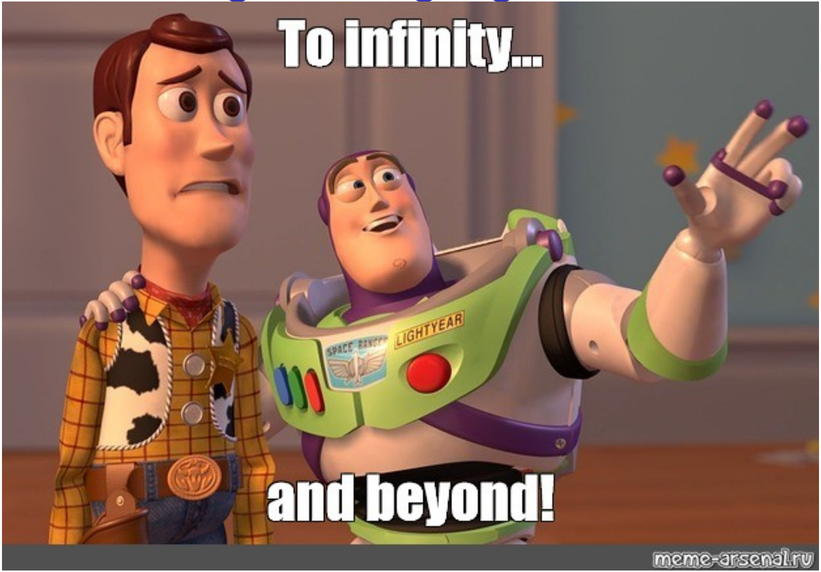
- What do we want to show for L^c?
  - That L^c is also regular...
  - i.e., there is also a DFA to recognize L^c

Closure of regular languages under complement

Technique: Go through every single regular language one by one, and show that its complement is also regular

Closure of regular languages under co



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  - We simply start with D, and then we flip the accept and reject states.
- Now let's try to give an airtight proof!

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  - $\Sigma^c = \Sigma$ (same alphabet)
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  - $\mathrm{F^c = Q \setminus F}$ (flip the accept/reject states)

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Reversal of a language

- Let Σ be an alphabet, $L \subseteq \Sigma^*$ be a formal language
- Let $w \in \Sigma^*$ be a string. Then w^r is the *reversal* of w, i.e. all the characters of w backwards
- $L^r = \{w^r | w \in L\}$ is the *reversal* of L, i.e. the backwards version of all the strings in L

Reversal of a language

- $L = \{\text{w} \mid \text{w starts with a and ends with b}\}$

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- $L = \{\text{w} \mid \text{w is an even integer}\}$
  - $L^r = \{\text{w} \mid \text{w starts with 0, 2, 4, 6, 8}\}$

Reversal of a language

Let $\Sigma = \{0, 1\}$, and let $L = \{w | w \text{ starts with } 01\}$.
Which of the following strings are in L^r

A) 01

B) 1010

C) 0101

D) 1111110

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- Let $\Sigma = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \ldots, \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix} \right\}$

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- $\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \in B$: $425 + 301 = 726$

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- $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix} \notin B$: $119 + 041 \neq 150$

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Which of the following strings are in B?

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C. $\begin{bmatrix}9\\1\\0\end{bmatrix} \begin{bmatrix}9\\1\\0\end{bmatrix} \begin{bmatrix}9\\1\\0\end{bmatrix}$

B. $\begin{bmatrix}0\\7\\8\end{bmatrix} \begin{bmatrix}3\\7\\0\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix}$

D. $\begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}4\\5\\6\end{bmatrix} \begin{bmatrix}7\\8\\9\end{bmatrix}$

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- **Proof:** see board

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- That is, if L is regular, then L^r is regular
- You will prove this on a future homework

Perfect shuffle

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- The *perfect shuffle* of A and B is $\mathrm{L} = \{\mathrm{w} \mid \mathrm{w} = a_1 b_1 a_2 b_2 \ldots a_n b_n$ where $a_1 a_2 \ldots a_n \in A$ and $b_1 b_2 \ldots b_n \in B\}$

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- The odd characters form a string in A
- The even characters form a string in B

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- Let $A = \{0, 00, 000, \ldots\}$ (i.e. all 0's, no 1's)

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- Let $A = \{0, 00, 000, \dots\}$ (i.e. all 0's, no 1's)
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- $010100 \notin \mathrm{PERFECT\text{-}SHUFFLE(A, B)}$
  - $000 \in A$
  - $110 \notin \mathrm{B}$

Perfect shuffle example

Let $\Sigma = \{a, b\}$.
Let $A = \{w | w \text{ has an even number of a's}\}$
Let $B = \{w | w \text{ ends with b}\}$
Which of the following strings are in $\text{PERFECT-SHUFFLE}(A, B)$?

A) aababaaa
B) bababab b
C) baabbaabbb
D) aabab

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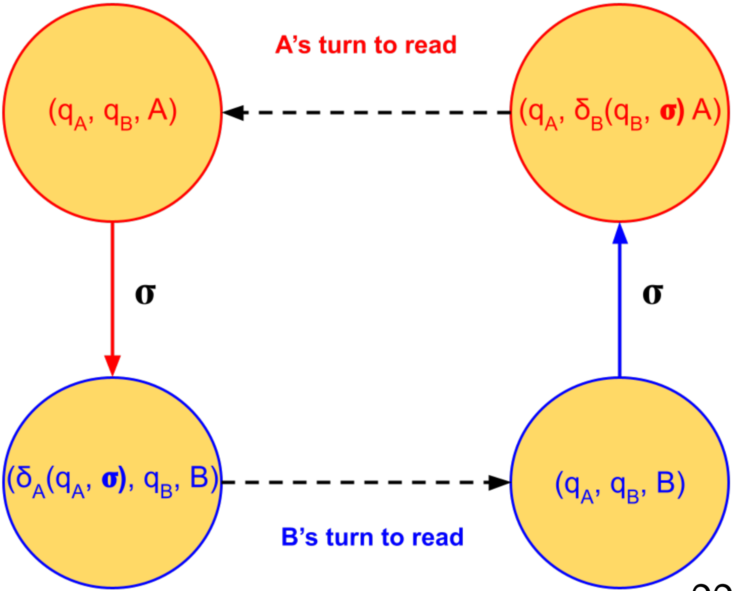

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  - A state $q_B \in Q_B$
  - A variable A or B that keeps track of turns

Perfect shuffle closure - transition function

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- $\delta((q_A, q_B, A), \sigma) = (\delta_A(q_A, \sigma), q_B, B)$
  - When it's A's turn, we transition A's state, keep B's state the same, and switch to B's turn to read

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- $\delta((q_A, q_B, B), \sigma) = (q_A, \delta_B(q_B, \sigma), A)$
  - When it's B's turn, we transition B's state, keep A's state the same, and switch to A's turn to read

Perfect shuffle closure - start state

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- We start out in A's start state

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- $q_s = (q_{s_A}, q_{s_B}, A)$
- We start out in A's start state
- We start out in B's start state
- Initially, it's A's turn to read

Perfect shuffle closure - accept states

Let $D_A = (Q_A, \Sigma, q_{s_A}, \delta_A, F_A)$ recognize A

Let $D_B = (Q_B, \Sigma, q_{s_B}, \delta_B, F_B)$ recognize B

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- A's state should be one of its accept states.

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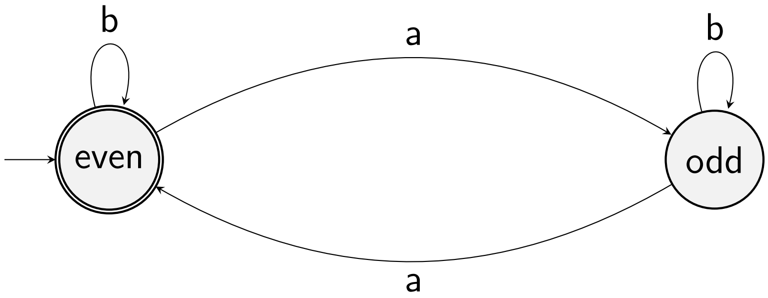
Perfect shuffle example

DFA for $A = \{w | w \text{ has an even number of a's}\}$



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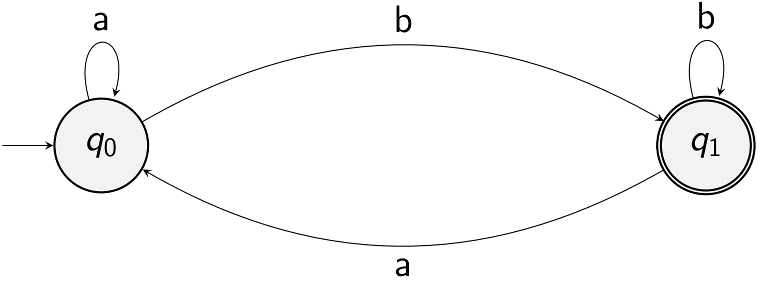

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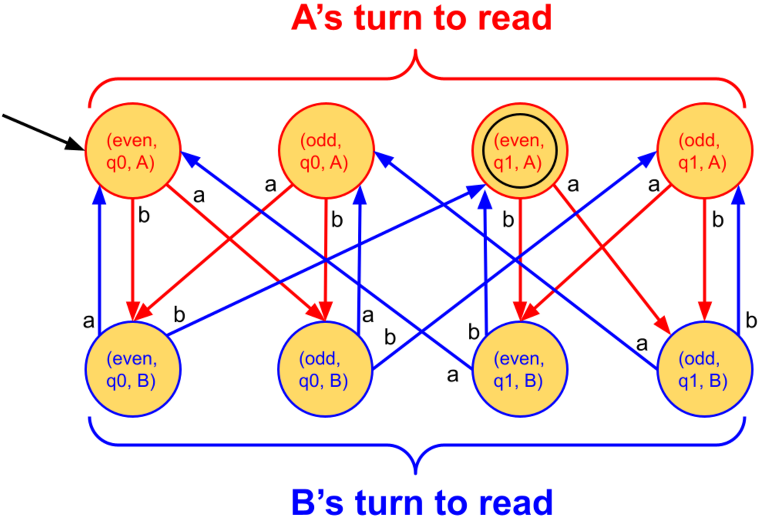
DFA for $\text{PERFECT-SHUFFLE}(A, B)$



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  - Note that A^* *always* includes empty string ϵ

Union Operation

Let $\Sigma = \{a, b\}$.
Let $A = \{w | w \text{ has an even number of a's}\}$
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Which of the following strings are in $A \cup B$?

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C) ab

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Kleene star operation

Let $\Sigma = \{a, b\}$.
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Which of the following strings are in B^*?

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A) ϵ ✓

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  - There exist DFAs D_1, D_2 that recognizes L_1 and L_2, respectively
- What do we want to show for $L_1 \cup L_2$?
  - Want to show that there is a DFA D_3 that recognizes $L_1 \cup L_2$

Closure of regular languages under union

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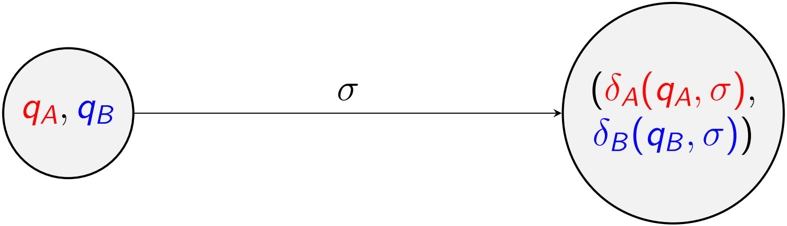
- **Proof idea:** Using D_1 and D_2, we will construct a DFA that runs both machines simultaneously and accepts if either machine accepts.

Closure of regular languages under union

- **Proof idea:** Using D_1 and D_2, we will construct a DFA that runs both machines simultaneously and accepts if either machine accepts.
- **Technique:** Run two DFAs *in parallel* using the cartesian product construction

Union idea



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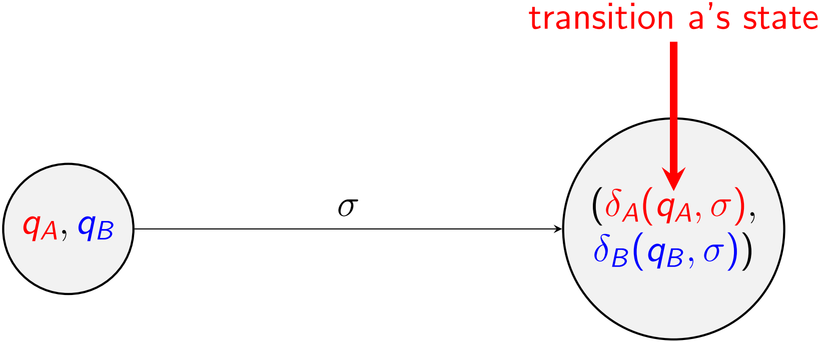
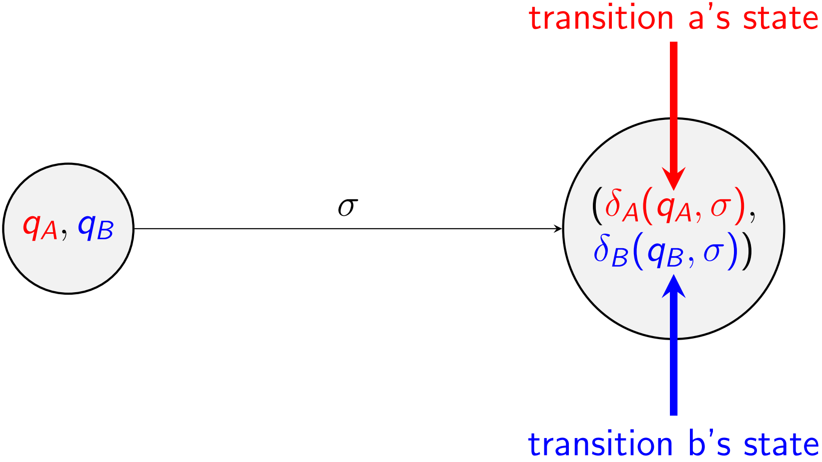

transition a's state
q_A, q_B
σ
$(\delta_A(q_A, \sigma),$
$\delta_B(q_B, \sigma))$

Union idea



transition a's state
q_A, q_B
σ
$(\delta_A(q_A, \sigma),$
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transition b's state

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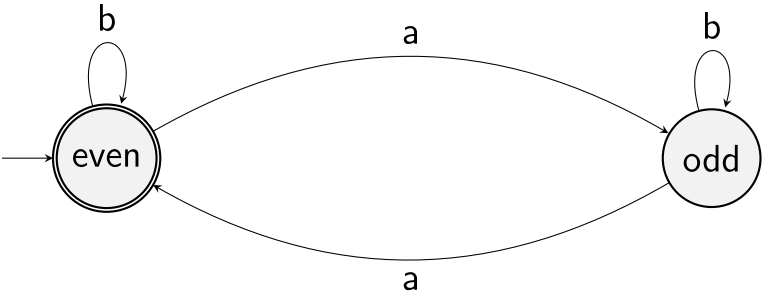
- ... or B's state should be one of its accept states
- (or perhaps both!)

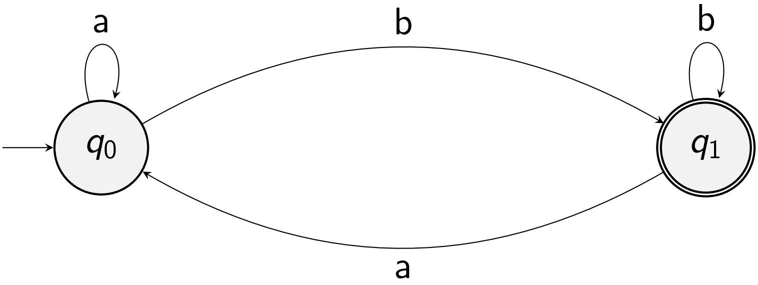
Union example

DFA for $A = \{w | w \text{ has an even number of a's}\}$

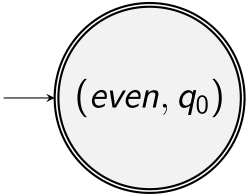


Union example

DFA for $B = \{w | w \text{ ends with b}\}$



Union example



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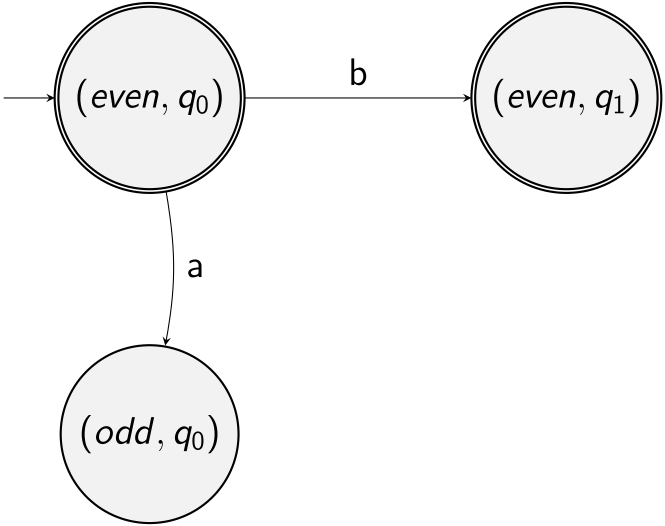

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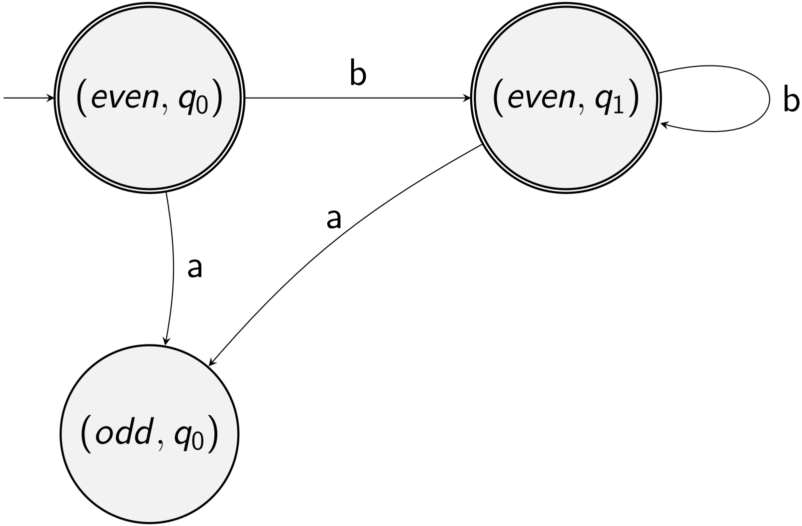

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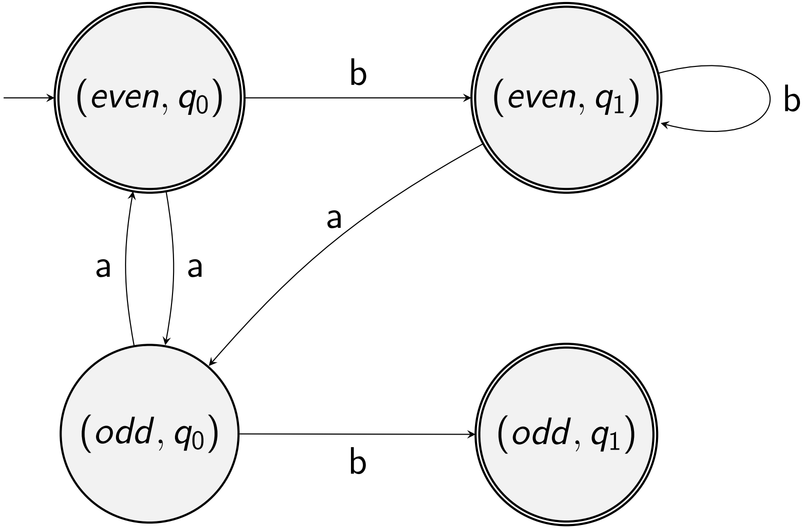

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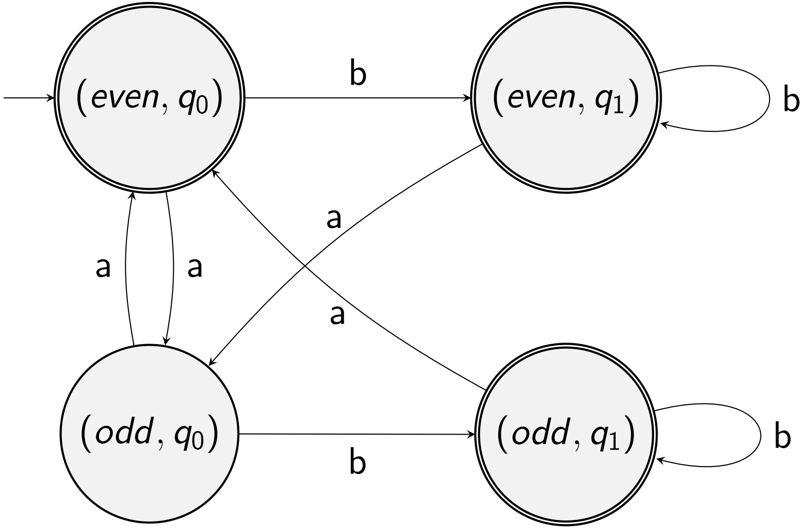

Closure of regular languages under intersection

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- Q: What technique do we use to do this?
  - **Technique 1:** Run the two machines in parallel using the Cartesian product construction
  - **Technique 2:** Express intersection in terms of other operations that we know regular languages are closed under

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  - Need an accept state for A *and* for B

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- Regular languages are closed under complement, so $(L_1^c \cup L_2^c)^c$ is regular
- Thus, $L_1 \cap L_2$ is regular!

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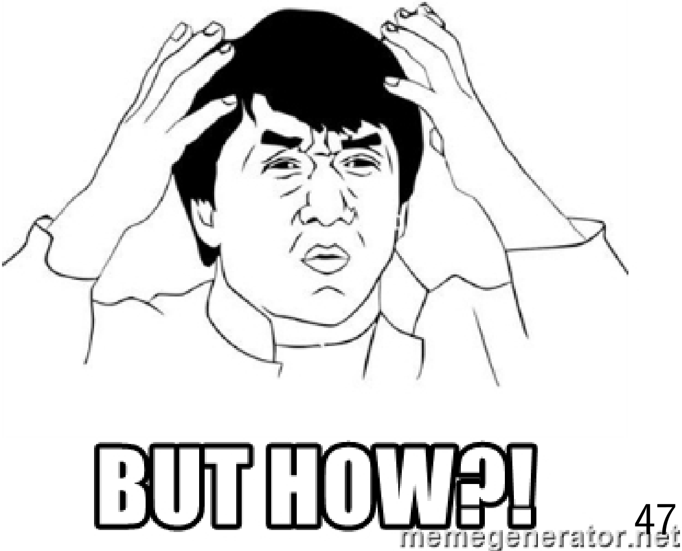

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Technique: nondeterminism!