Decision problems, formal languages, and compuation

Arjun Chandrasekhar

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Can we design a machine to produce the correct output every time?

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Are decision problems and function problems the equivalent?

Decision vs Function problem

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  - Find the shortest path length, check if it is $\leq k$

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  - Yes!
  - Let n be the total number of vertices in G
  - For $k = 1, 2, \ldots n$: check if the shortest path length is $\leq k$
  - Stop the first time we receive an output of ACCEPT

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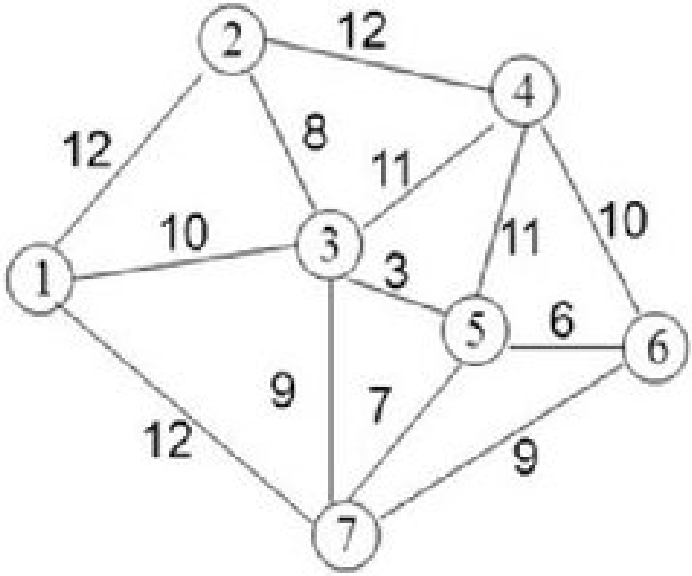
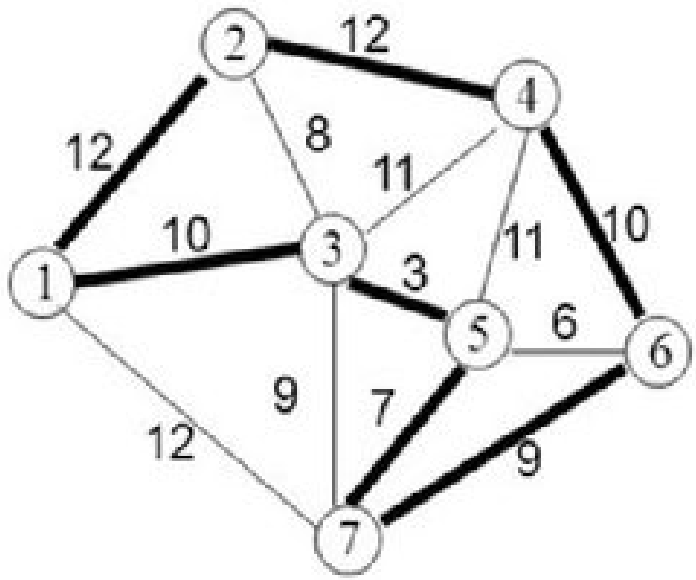


a)



b)

Travelling Salesman Problem

Travelling Salesman Problem: try find a path that visits all nodes in a graph using the lowest possible edge weights

1. What is the function problem associated with travelling salesman?
2. What is the decision problem associated with travelling salesman?
3. If we can solve the function problem, how can we solve the decision problem?
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- **Input:** A graph G
- **Output:** The length of (one of) the shortest path(s) that includes every vertex

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  - Check if there is a path of length $\leq k$ that includes each vertex
  - If yes, then output k. Otherwise, keep searching

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- **Def:** A **string** is a sequence of symbols from some alphabet
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- Let Σ be an alphabet. Then Σ^* is the set of all possible strings on that alphabet

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Note: ϵ refers to the empty string

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Can we design computers to solve these decision problems?

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- $L_5 = \{w \mid w \text{ is a valid encoding for the shortest path decision problem on a graph for which the answer is yes}\}$

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  - Infinite Loop

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