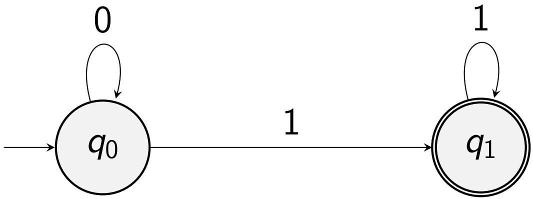
Nondeterministic Finite Automata

Arjun Chandrasekhar

Nondeterministic Finite Automata

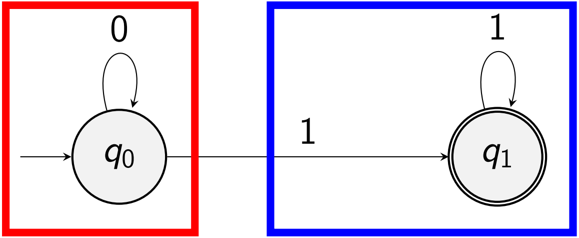
- A DFA is *deterministic*. For each state/symbol combination, there is exactly one transition defined.
- An **nondeterministic finite automaton (NFA)** is like a DFA, except a state/symbol pair may have any number of transitions defined for it (0, 1, 2, ...).
- Can also have ϵ transitions which let you change states without reading a symbol.

Nondeterministic Finite Automata



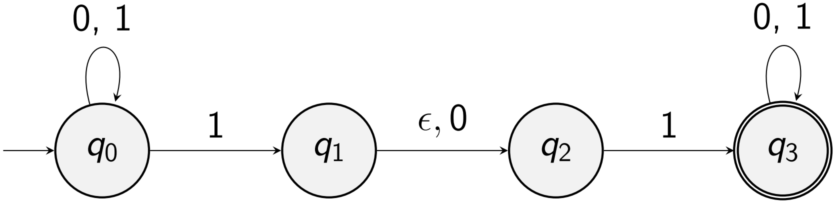
Nondeterministic Finite Automata

Can only read 0s here



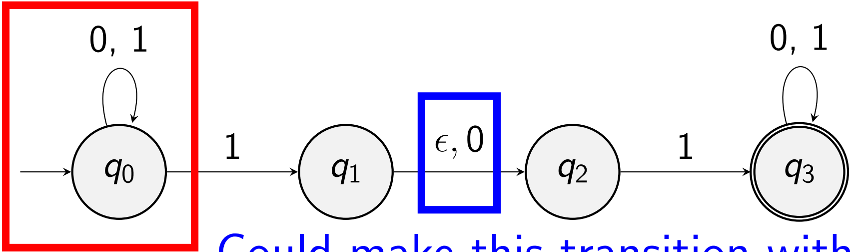
Can only read 1s here

Nondeterministic Finite Automata



Nondeterministic Finite Automata

Multiple transitions for the same symbol



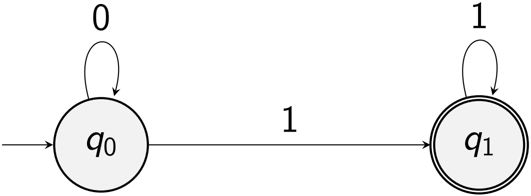
Could make this transition with or without reading a symbol

Computation on an NFA

- Start in the start state
- Scan symbols one-by-one
- For each symbol σ scanned:
  - Go to *one of* the possible arrows with the label σ
  - If no arrows have the label σ the computation *dies*
  - The NFA can behave in different ways on the same input string!
- At any point the NFA may take an ϵ transition without consuming a character
- The NFA accepts if after reading all the characters, and taking any desired ϵ transitions, it is in an accept state

Computation on an NFA

What happens on inputs: 000, 010, 101, 011?



$$q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \rightarrow \text{REJECT}$$

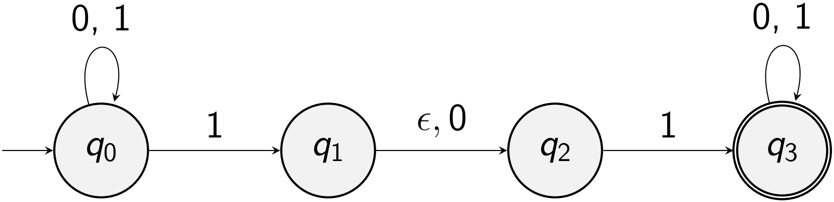
$$q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_1 \xrightarrow{0} \text{DIES}$$

$$q_0 \xrightarrow{1} q_1 \xrightarrow{0} \text{DIES}$$

$$q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_1 \xrightarrow{1} q_1 \rightarrow \text{ACCEPT}$$

Computation on an NFA

What happens on input 111?



$$q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_0 \rightarrow \text{REJECT}$$

$$q_0 \xrightarrow{1} q_1 \xrightarrow{1} \text{DIES}$$

$$q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_0 \xrightarrow{1} q_1 \xrightarrow{\epsilon} q_2 \rightarrow \text{REJECT}$$

$$q_0 \xrightarrow{1} q_1 \xrightarrow{\epsilon} q_2 \xrightarrow{1} q_3 \xrightarrow{1} q_3 \rightarrow \text{ACCEPT}$$

NFA Formal Definition

Def: A **Nondeterministic finite automate (NFA)** is a 5-tuple $(Q, \Sigma, \delta, q_s, F)$

- Q: The set of states in the NFA
- Σ the alphabet of (non-ϵ) characters that the NFA can read
- q_s: the starting state
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to \mathcal{P}(Q)$ - the transition function
  - **Input:** Current state & next symbol (or ϵ)
  - **Output:** *Set* of possible next states (could be empty)
- F - the set of accept states

NFA Accepting Computation

- An NFA can do many different things on the same string
  - It may be capable of both accepting *and* rejecting the same string!
- What does it mean for an NFA to accept a string?
- Informally, an NFA accepts a string w if there *exists* a computation path that ends in an accept state
  - Even if every other path rejects and/or dies, just one accepting path is good enough

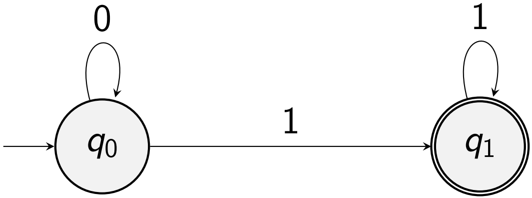
NFA Accepting Computation

An NFA accepts a string $w = w_1 w_2 \ldots w_n$ if:

1. We can re-write w as $y = y_1 y_2 \ldots y_n$ where each $y_i \in (\Sigma \cup \epsilon)$ (i.e. insert empty ϵ characters into w) and ...
2. There *exists* a sequence of states $q_0 q_1 \ldots q_n$ such that...
   2.1 $q_0 = q_s$ (start in the start state)
   2.2 $q_i \in \delta(q_{i-1}, y_i)$ for all i (all transitions are valid)
   2.3 $q_n \in F$ (end in an accept state)

NFA Accepting Computation

Which strings are accepted by this NFA?



A) ϵ (empty string)

B) 1

C) 010

D) 101

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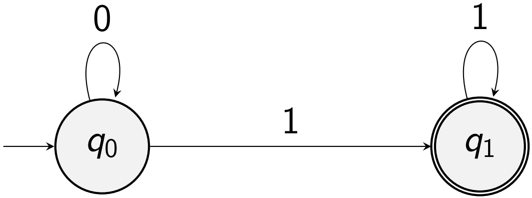

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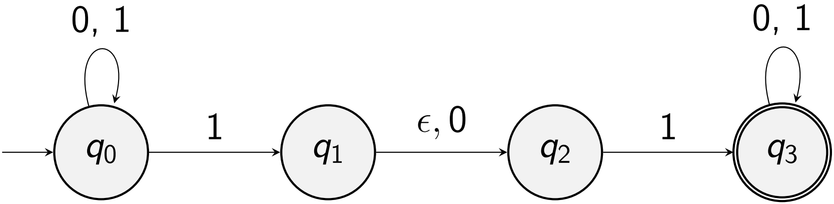


A) ϵ (empty string)

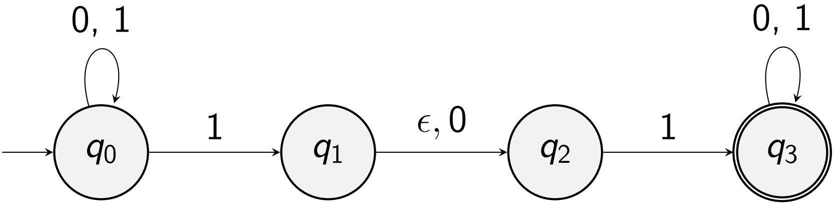
B) 111

C) 111101000

D) 0000

NFA Accepting Computation

Which strings are accepted by this NFA?



A) ϵ (empty string)

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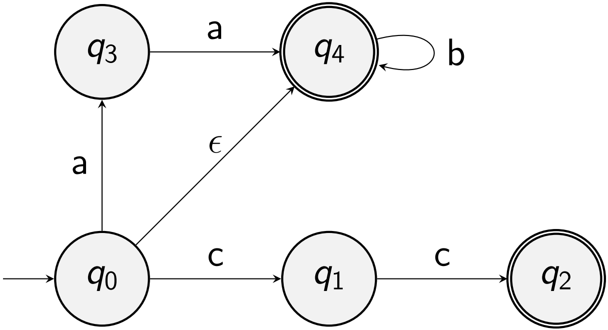

A) ϵ (empty string)

B) *abba*

C) *cc*

D) *cccccccccccc*

NFA Accepting Computation

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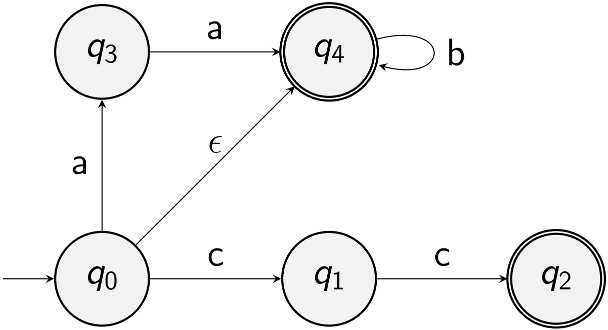


A) ϵ (empty string) ✓

B) *abba*

C) *cc* ✓

D) *cccccccccccc*

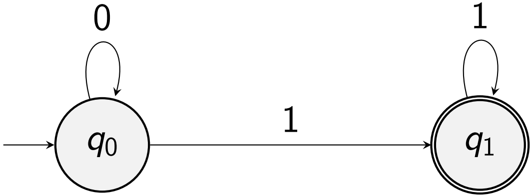
The Language of an NFA

- Let N be an NFA
- The *language of N* is the set of strings that N accepts i.e.

$$L(N) = \{w | N \text{ accepts } w\}$$

The Language of an NFA

What is the language of this NFA



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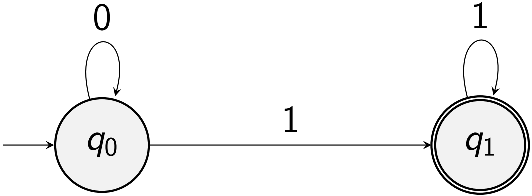

$L(N) = \{w \mid \text{0s precede 1s, at least one 1}\}$

The Language of an NFA

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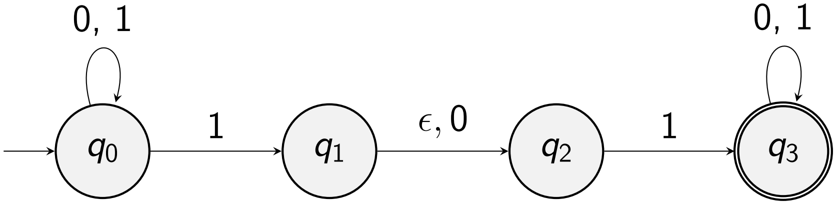

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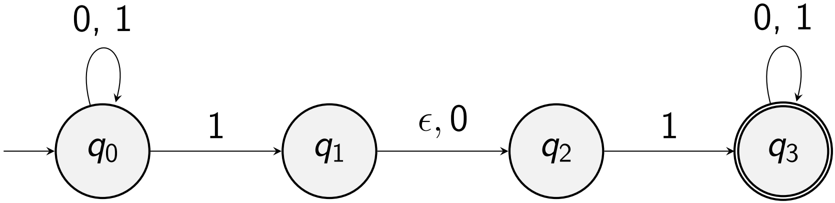
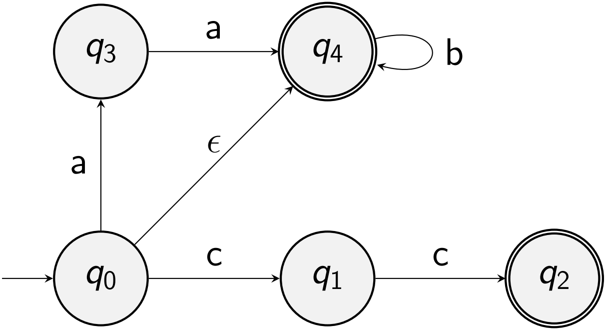


L(N) = {w | w contains 101 or 11 as a substring}

The Language of an NFA

What is the language of this NFA



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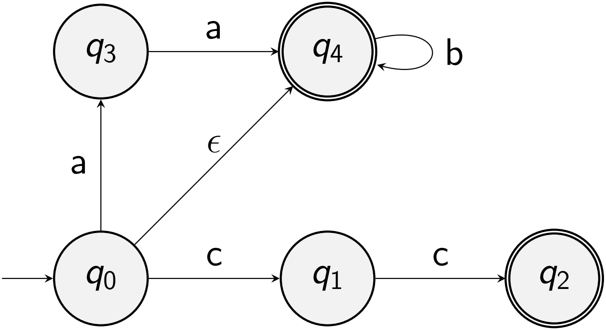

L(N) = {w | w has either zero or two a's followed by any number of b's, OR w = cc}

Nondeterminism

- As said earlier, an NFA can have many possible computation paths
- We can think of nondeterminism in two ways:
  - The NFA "guesses" which choice will ultimately lead to an accepting state
  - The NFA branches/copies itself for each possible choice.

NFAs vs DFAs

- Are NFAs more powerful than DFAs?
  - That is, are there languages that an NFA can recognize, but a DFA cannot?
- As it turns out, no! So why study them?
  - If we want to show a langauge is regular, It is often easier to describe an NFA than a DFA.
  - If we actually want to be able to recognize the language, then we can automate the conversion of an NFA to a DFA.

Designing an NFA

Design a 4-state NFA to recognize the following language: $L = \{w \mid w \text{ ends with bba}\}$

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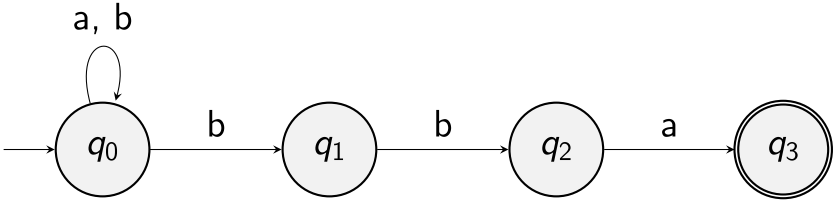
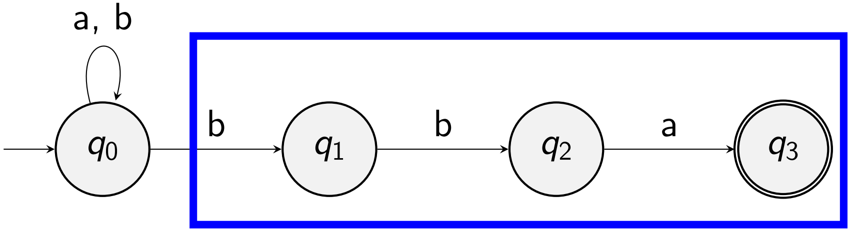


Designing an NFA

Design a 4-state NFA to recognize the following language: $L = \{w \mid w \text{ ends with bba}\}$

"Guess" when we've reached the end



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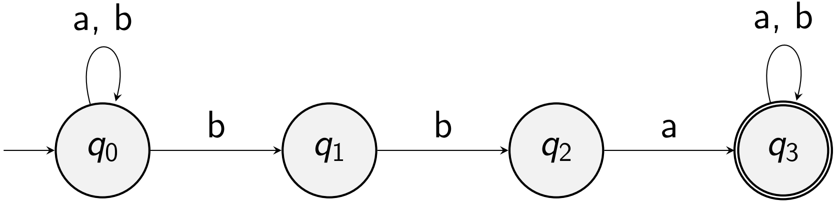
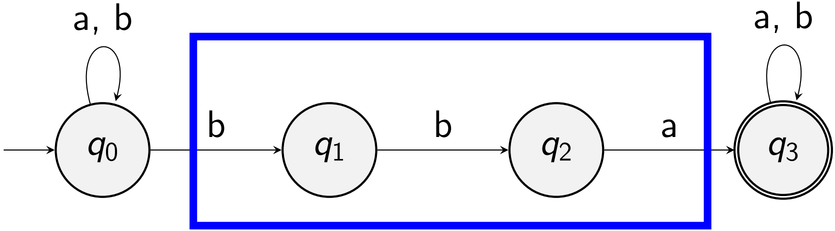


Designing an NFA

Design a 4-state NFA to recognize the following language: L = {w | w contains bba}

"Guess" where bba occurs



Combining NFAs

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Let $\Sigma = \{0\}$. Design an NFA to recognize strings with an even number of 0s

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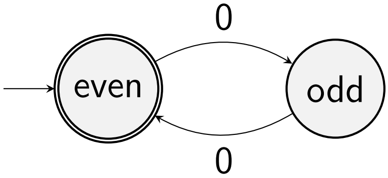

Combining NFAs

Let $\Sigma = \{0\}$. Design an NFA to recognize strings with an exactly three 0s

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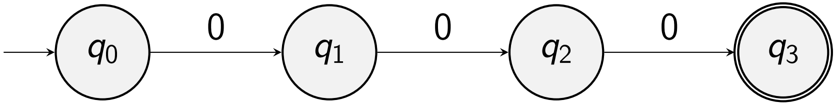

Combining NFAs

Let $\Sigma = \{0\}$. Design an NFA to recognize strings where the number of 0s is even or exactly 3

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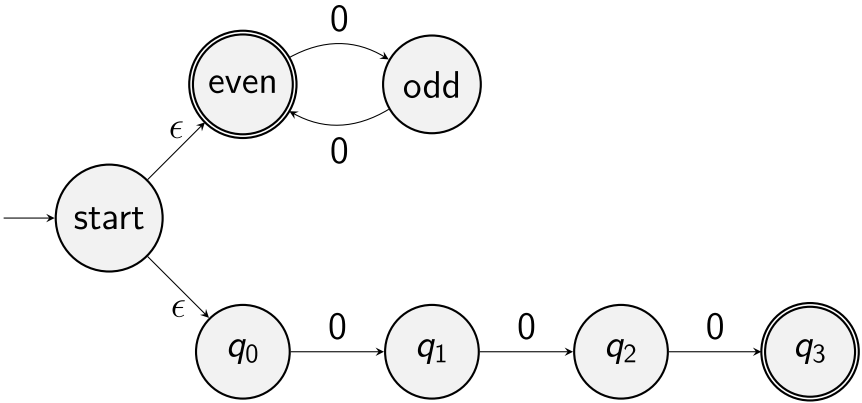

Combining NFAs

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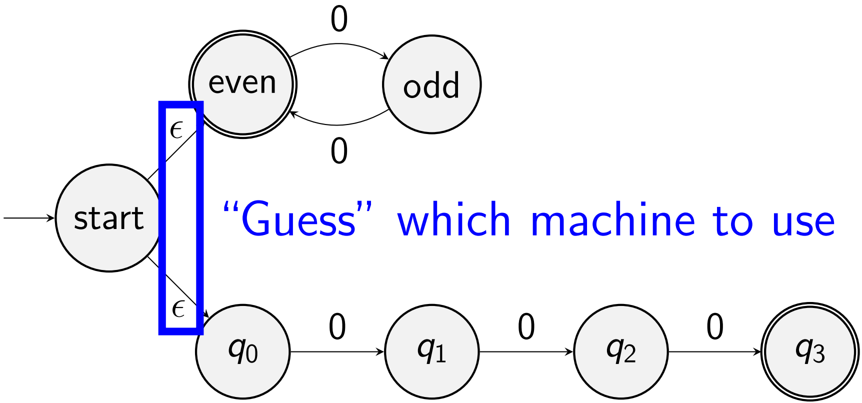


Equivalence of NFAs and DFAs

Theorem: A language is recognized by an NFA if and only if it is recognized by a DFA

- **Proof idea:** We will show that every NFA N can be converted to an equivalent DFA D that recognizes all the same strings
- **Technique:** Simulate nondeterminism using the power set construction
  - Every state in the D will correspond to a *subset of states* in N, i.e. set of possible states where N *could be* at some point in the computation
  - Every transition in D will correspond to *all* of the possible states N could reach from *any* of the states in the previous step
  - Accept if the NFA *could be* in an accept state

Equivalence between NFAs and DFAs

(⇒) If a language L is recognized by a DFA, then there exists an NFA to recognize it

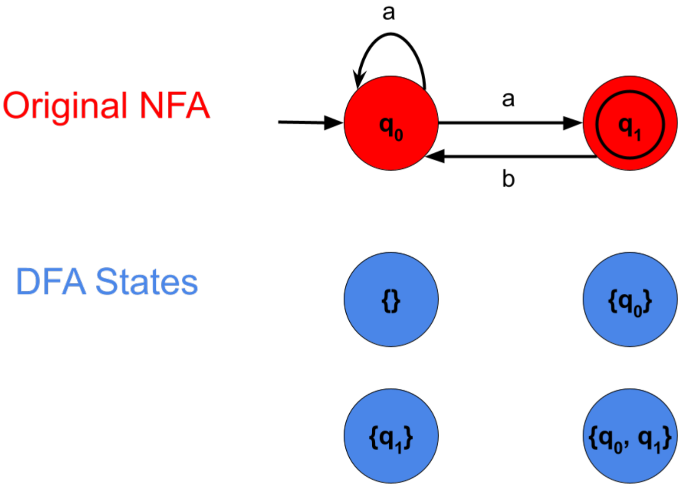
- Suppose there is a DFA D that recognizes L
- Then D *is* an NFA!
  - It's an NFA that simply chooses not to have any nondeterminism, missing transitions, or ϵ transition
- Thus, there exists an NFA that recognizes L

Equivalence between NFAs and DFAs

($\Leftarrow$) If a language L is recognized by an NFA, then there exists a DFA to recognize it

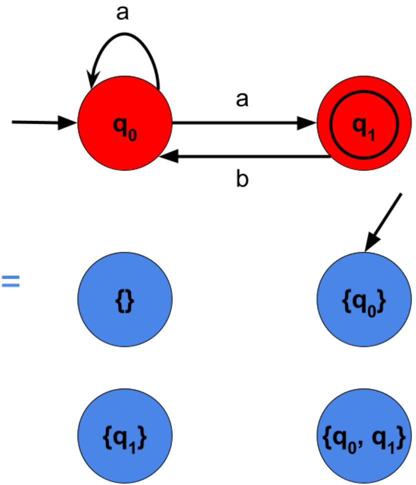
- Suppose there is an NFA $N = (Q_N, \Sigma, q_{s_N}, \delta_N, F_N)$ that recognizes L
- For now, assume N has no ϵ transitions
- We will construct a DFA $D = (Q_D, \Sigma, q_{S_D}, \delta_D, F_D)$ to recognize L
  - $Q_D = \mathcal{P}(Q_N)$
  - $\delta_D(R, \sigma) = \bigcup_{r \in R} \delta_N(r, \sigma)$
  - $q_{S_D} = \{q_{s_N}\}$
  - $F_D = \{R \subseteq Q_N | R \cap F_N \neq \emptyset\}$ (i.e., all subsets that include at least one accept state)

NFA to DFA conversion



NFA to DFA conversion

Original NFA
start state = q_0



DFA start state = $\{q_0\}$

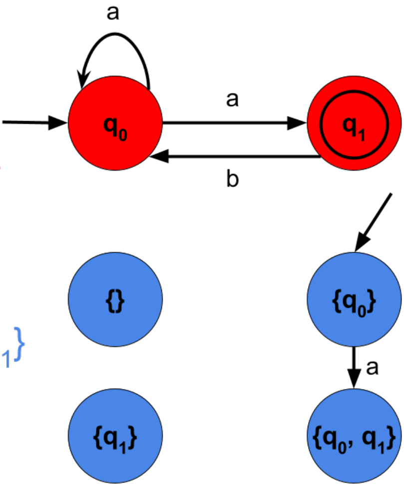
NFA to DFA conversion

Original NFA

$\delta(q_0, a) = \{q_0, q_1\}$

DFA

$\delta(\{q_0\}, a) = \{q_0, q_1\}$



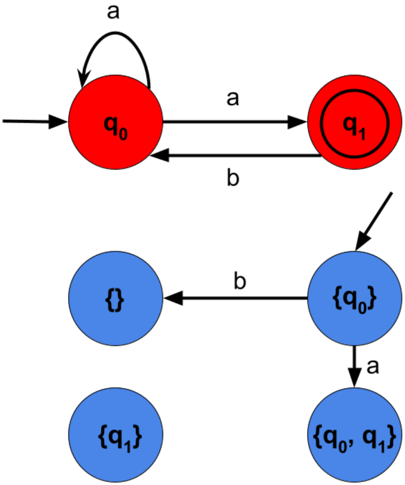
NFA to DFA conversion

Original NFA

$\delta(q_0, b) = \{\}$

DFA

$\delta(\{q_0\}, b) = \{\}$



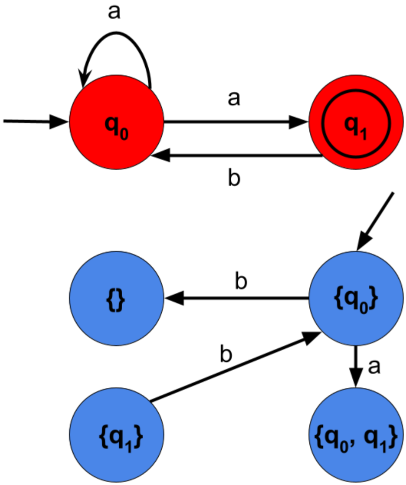
NFA to DFA conversion

Original NFA

$\delta(q_1, b) = \{q_0\}$

DFA

$\delta(\{q_1\}, b) = \{q_0\}$



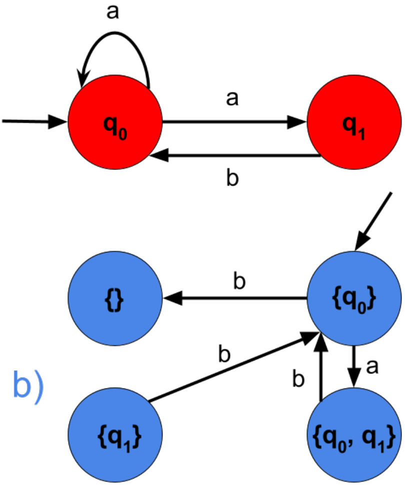
NFA to DFA conversion

Original NFA

$\delta(q_0, b) = \{\}$
$\delta(q_1, b) = \{q_0\}$

DFA

$\delta(\{q_0, q_1\}, b)$
$= \delta(q_0, b) \cup \delta(q_1, b)$
$= \{\} \cup \{q_0\}$
$= \{q_0\}$



NFA to DFA conversion

Original NFA

$\delta(q_0, a) = \{q_0, q_1\}$
$\delta(q_1, a) = \{\}$

DFA

$\delta(\{q_0, q_1\}, a)$
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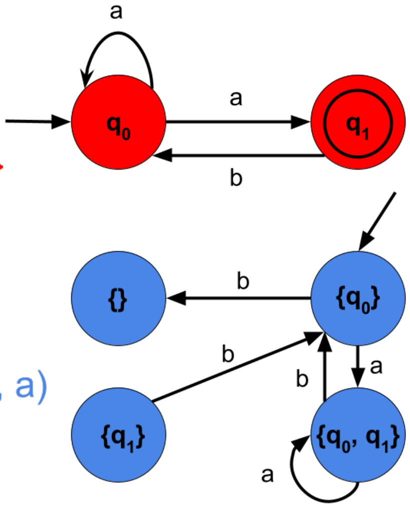
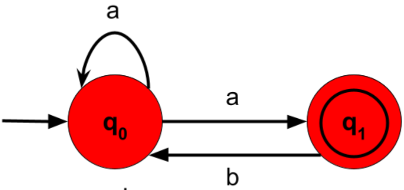
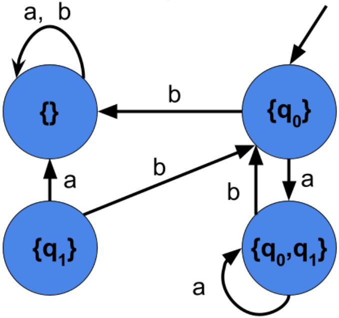


NFA to DFA conversion

Original NFA



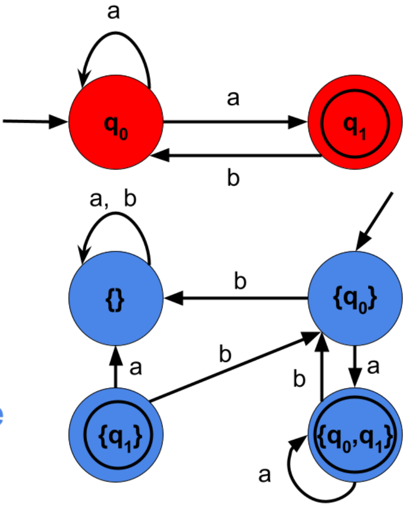
DFA with all transitions



NFA to DFA conversion

Original NFA
F = $\{q_1\}$

DFA
F = anything containing an NFA accept state
= $\{\{q_1\}, \{q_0, q_1\}\}$



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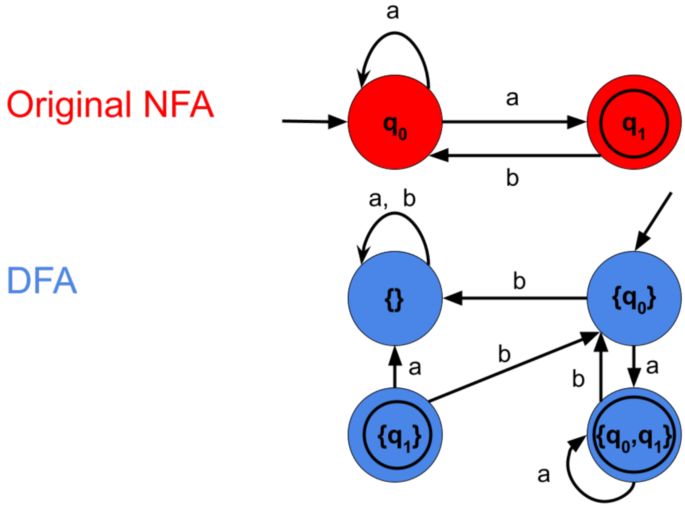
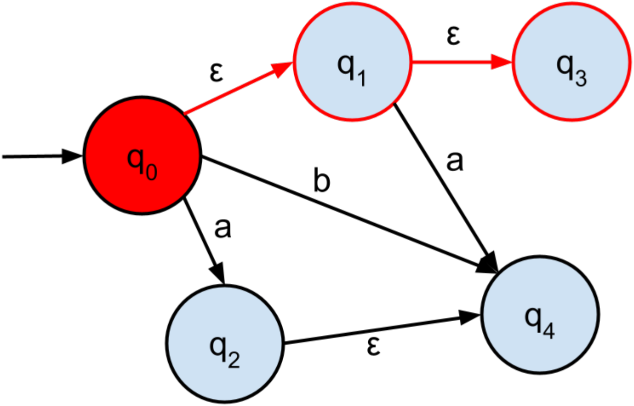


Epsilon Closure

- Let $N = (Q, \Sigma, q_s, \delta, F)$ be an NFA
- Let $S \subseteq Q$ be a set of states
- **Def:** the **epsilon closure** $E(S)$ is the set of states that can be reached from S using only ϵ arrows
  - This includes members of S

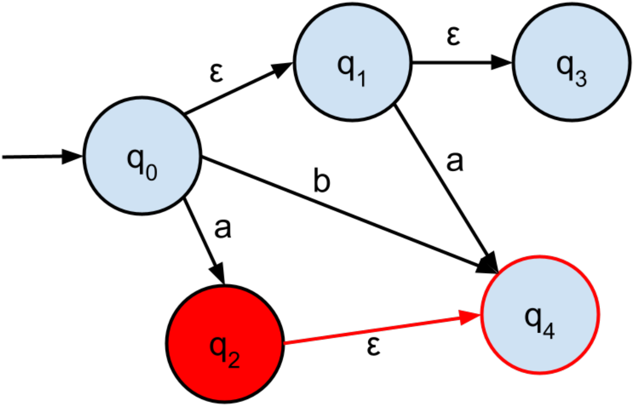
Epsilon Closure Example

$E(\{q_0\}) = \{q_0, q_1, q_3\}$



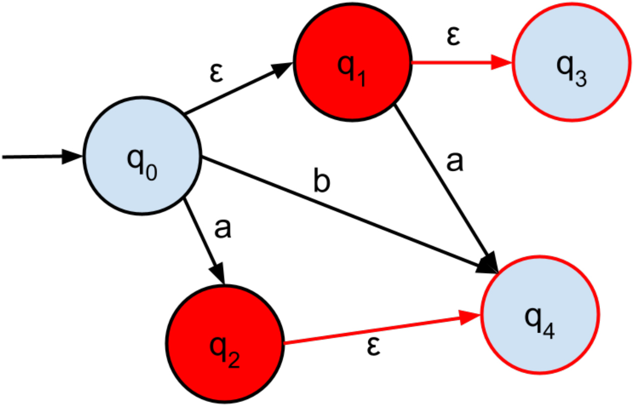
Epsilon Closure Example

$E(\{q_2\}) = \{q_2, q_4\}$



Epsilon Closure Example

$$E(\{q_1, q_2\}) = \{q_1, q_2, q_3, q_4\}$$



NFA to DFA conversion

How do we extend our conversion to account for ϵ transitions?

- $Q = \mathcal{P}(Q_N)$
- $\delta_D(R, \sigma) = \bigcup_{r \in R} \delta_N(r, \sigma)$
- $q_{S_D} = \{q_{s_N}\}$
- $F_D = \{R \subseteq Q_N | R \cap F_N \neq \emptyset\}$

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NFA to DFA conversion

How do we extend our conversion to account for ϵ transitions?

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- $\delta_D(R, \sigma) = E\left(\bigcup_{r \in R} \delta_N(r, \sigma)\right)$
- $q_{S_D} = E(\{q_{s_N}\})$
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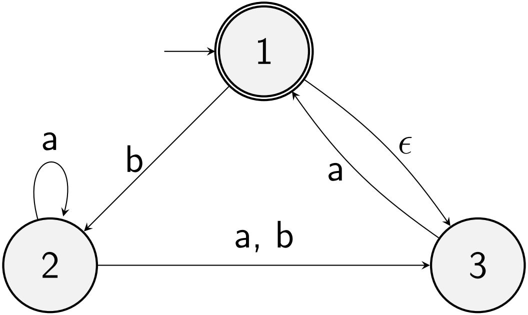
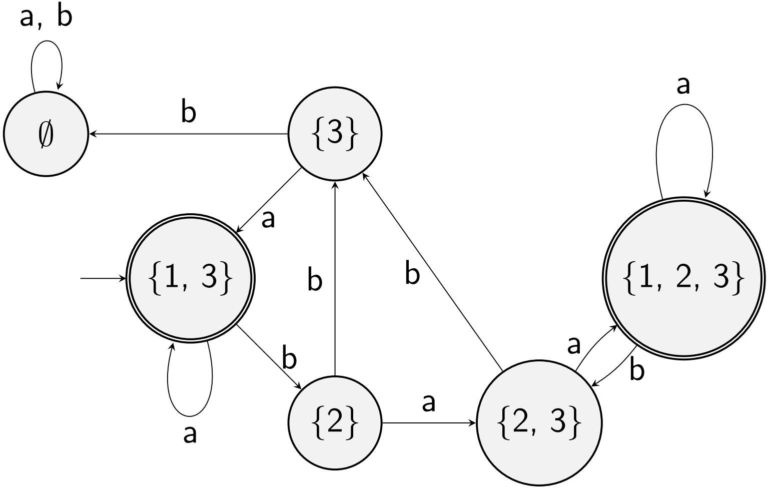
NFA to DFA Conversion Example

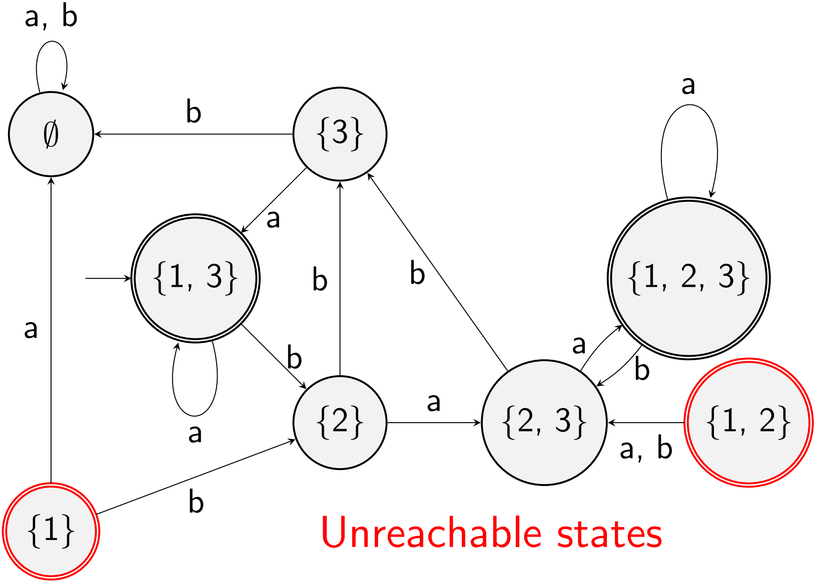
Let's convert the following NFA to a DFA



NFA to DFA Conversion Example



NFA to DFA Conversion Example



NFAs and regular languages

- Recall that the regular languages are the languages recognized by DFAs
- We have proven that DFAs and NFAs are equivalent

NFAs and regular languages

NFAs and regular languages

- Recall that the regular languages are the languages recognized by DFAs
- We have proven that DFAs and NFAs are equivalent
- **Corollary:** a language is regular if and only if it is recognized by an NFA
- It will often be more convenient use NFAs when we want to show that a langauge is regular!

Regular operations

Recall the regular operations:

- **Union:**
  $A \cup B = \{w | w \in A \text{ or } w \in B\}$
- **Concatenation:**
  $A \circ B = \{w = w_1 w_2 | w_1 \in A, w_2 \in B\}$
- **(Kleene) Star:**
  $A^* = \{\epsilon\} \cup \{w = w_1 w_2 \ldots w_n | w_i \in A\}$

Kleene's Theorem

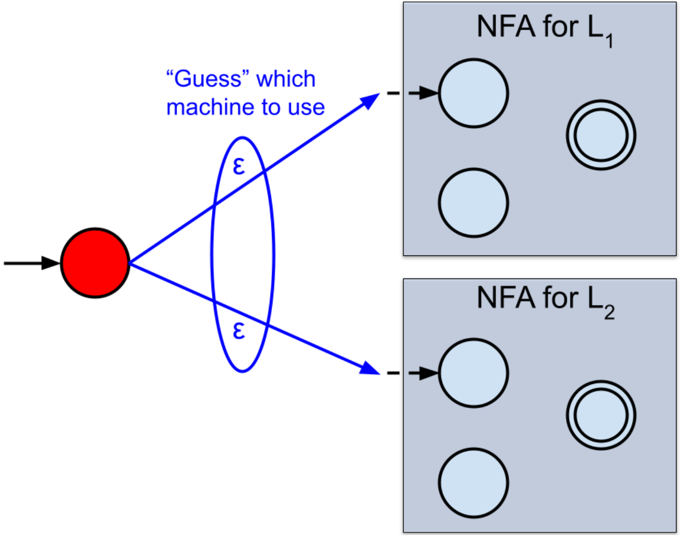
Theorem: The regular languages are closed under the regular operations

- Want to show that if L_1 and L_2 are regular, then $L_1 \cup L_2$, $L_1 \circ L_2$, and L_1^* are regular
- With DFAs, it was messy
- With NFAs, this will be easy!
- **Proof idea:** We will combine the DFAs for L_1 and L_2 into an NFA that simulates the regular operation.
  - For Kleene star we only modify the DFA for L_1

Closure under union

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add a new start state
- Add ϵ transitions to the two original start states

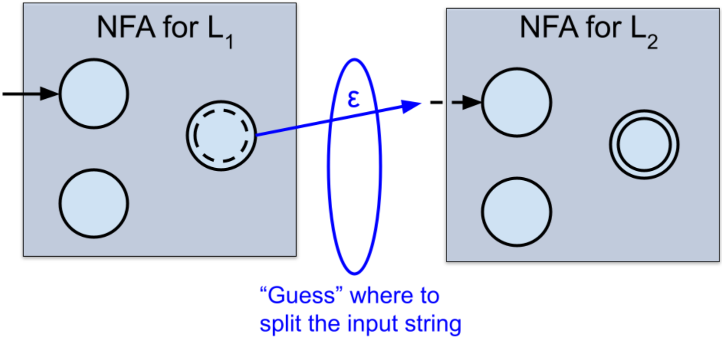
Closure under union



Closure under concatenation

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add an ϵ transition between N_1's accept state(s) and N_2's start state
- Accept states in N_1 are no longer accept states (we have to accept in N_2)

Closure under concatenation



Closure under Kleene star

- Let N_1 recognize L_1
- Start with the smaller NFA
- Add ϵ transitions from each accept state back to the start state
- Add an new start state with an ϵ transition to the original start state
  - This new start state will also be an accept state

Closure under Kleene star