Nondeterministic Finite Automata

Arjun Chandrasekhar

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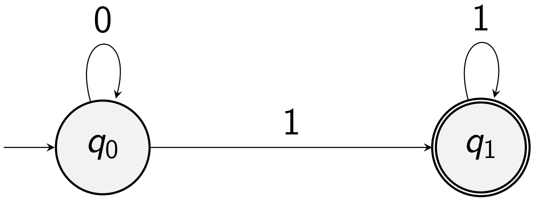
- Can also have ϵ transitions which let you change states without reading a symbol.

Nondeterministic Finite Automata



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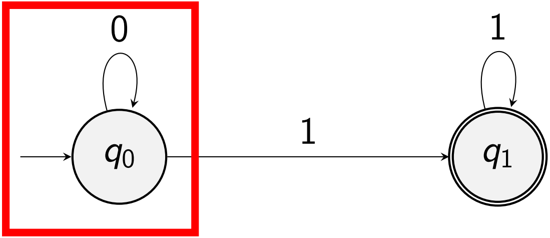

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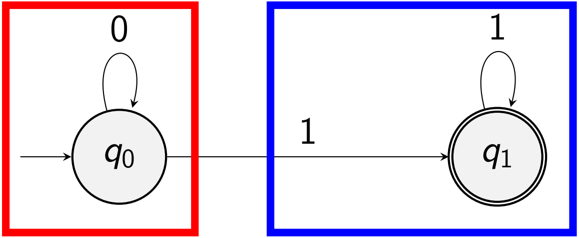

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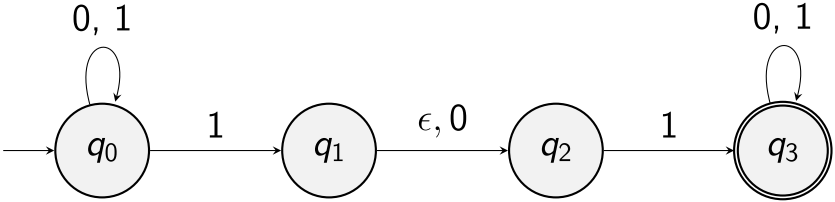

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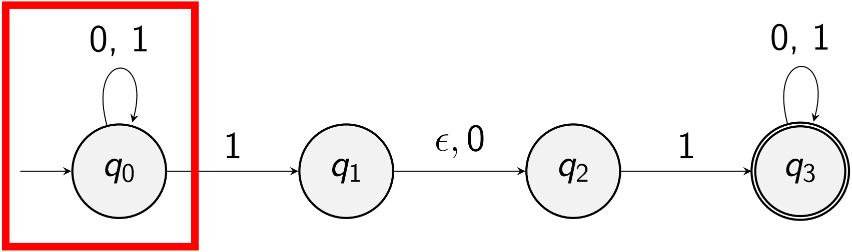

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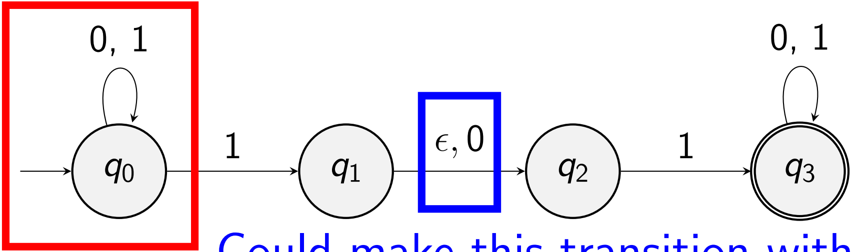

Could make this transition with
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- At any point the NFA may take an ϵ transition without consuming a character
- The NFA accepts if after reading all the characters, and taking any desired ϵ transitions, it is in an accept state

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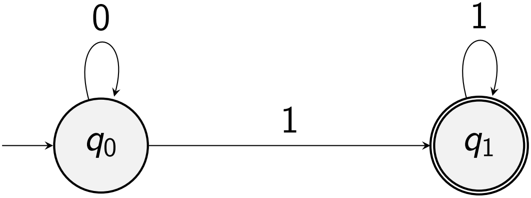
What happens on inputs: 000, 010, 101, 011?



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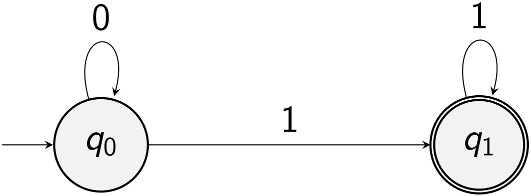

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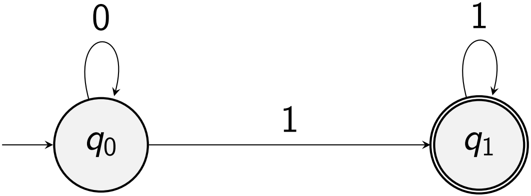

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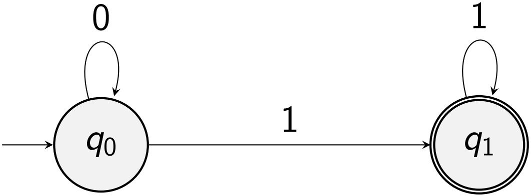

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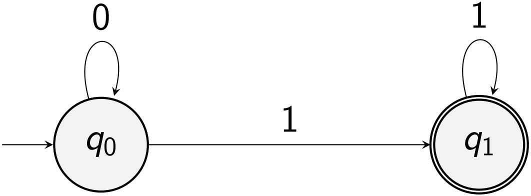

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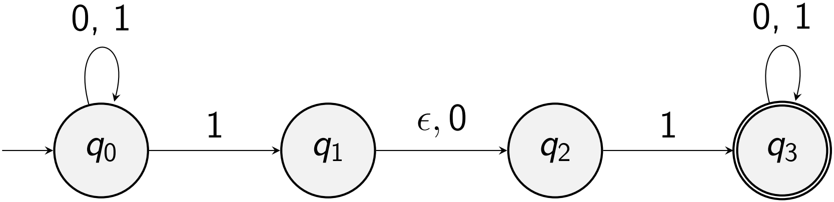

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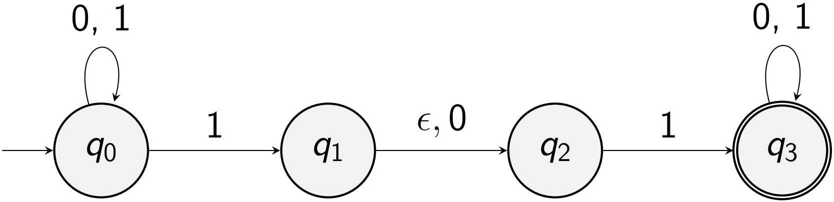

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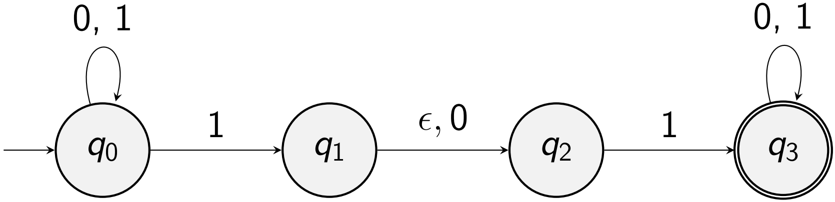

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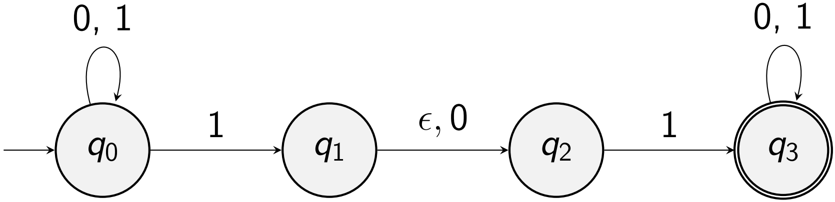

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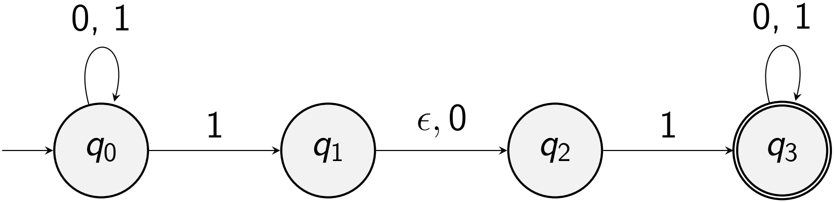

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  - Even if every other path rejects and/or dies, just one accepting path is good enough

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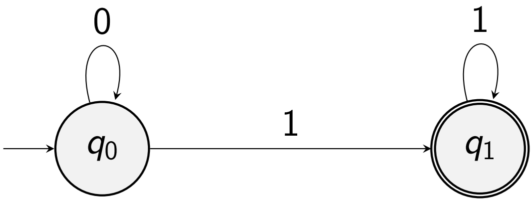

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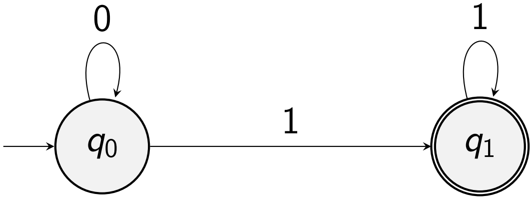

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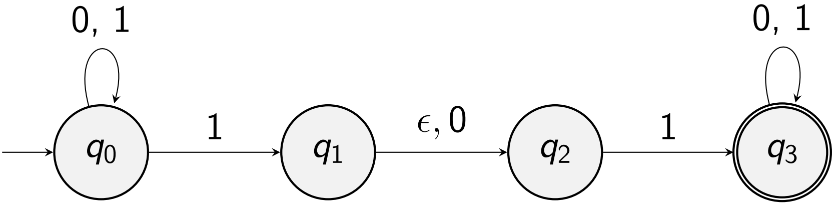

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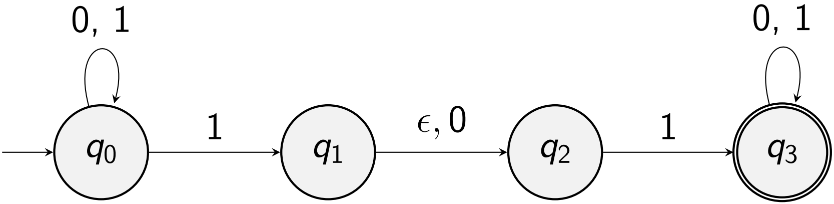

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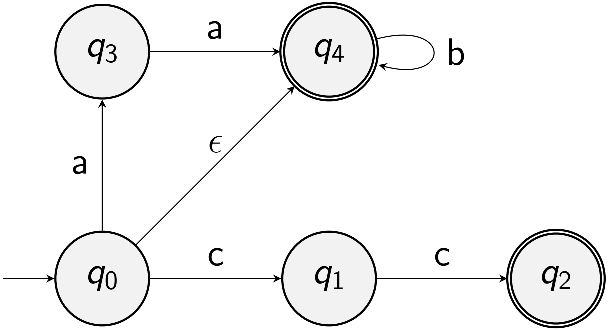

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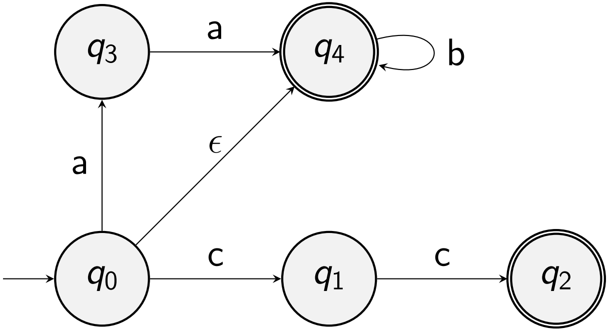

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- The *language of N* is the set of strings that N accepts i.e.

$$L(N) = \{w | N \text{ accepts } w\}$$

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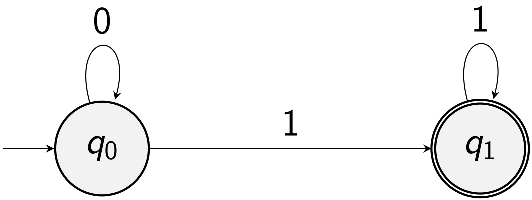

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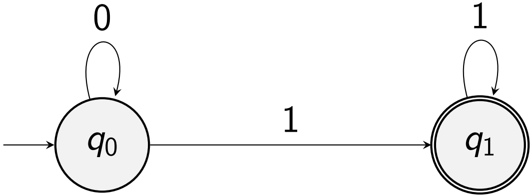

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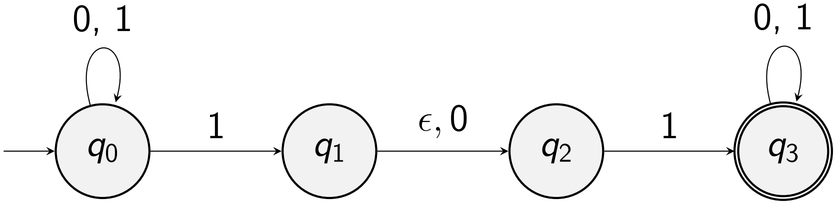

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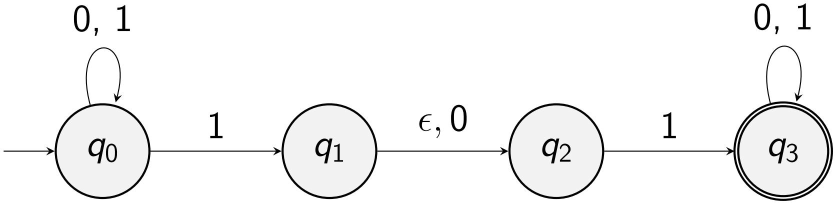

L(N) = {w | w contains 101 or 11 as a substring}

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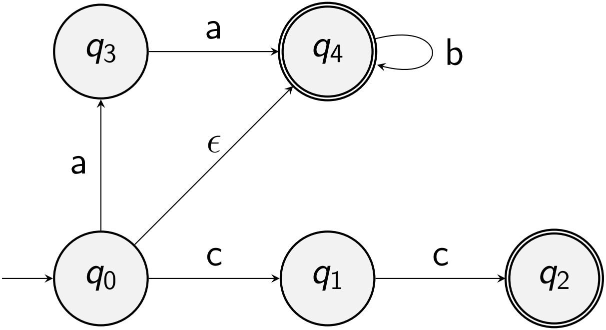

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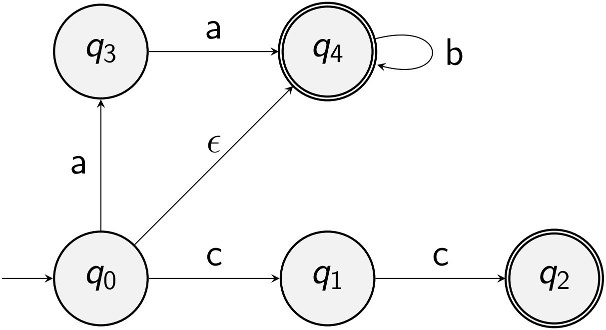

L(N) = {w | w has either zero or two a's followed by any number of b's, OR w = cc}

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  - The NFA branches/copies itself for each possible choice.

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  - If we want to show a langauge is regular, It is often easier to describe an NFA than a DFA.
  - If we actually want to be able to recognize the language, then we can automate the conversion of an NFA to a DFA.

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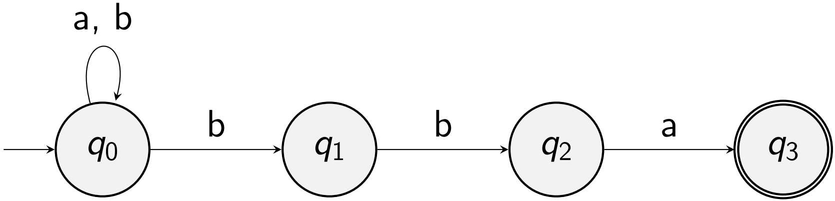

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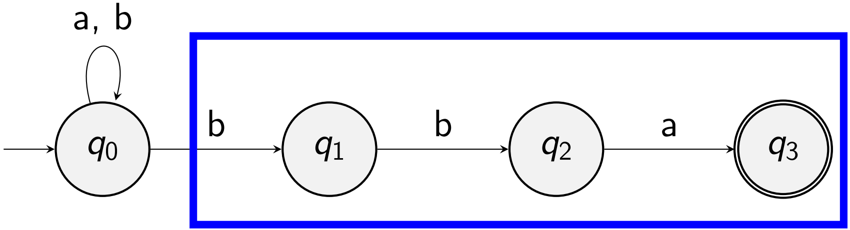
"Guess" when we've reached the end



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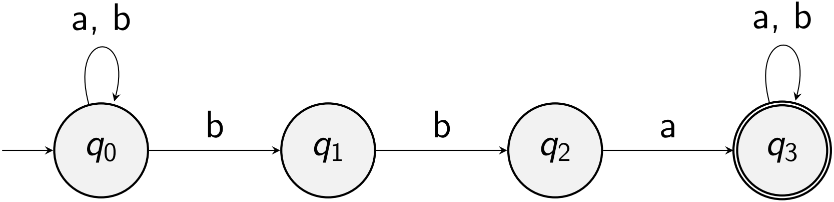

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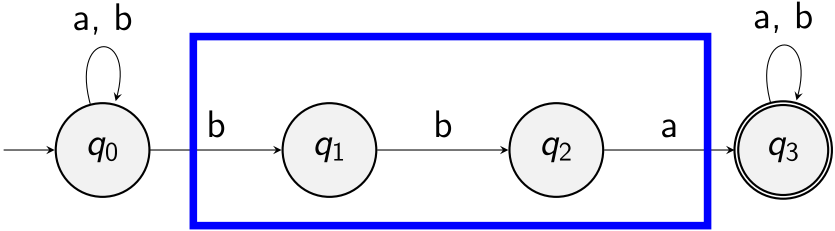
"Guess" where bba occurs



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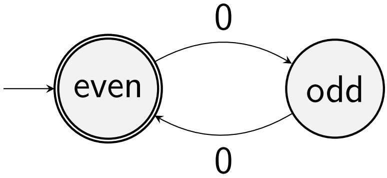

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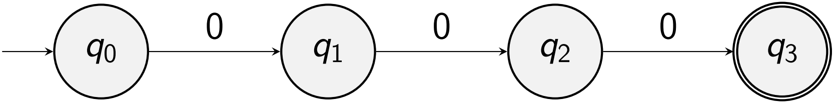

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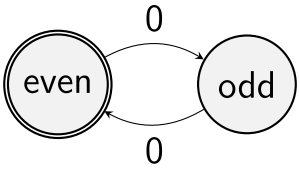

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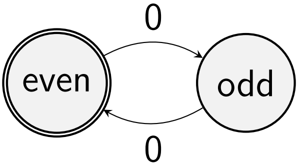

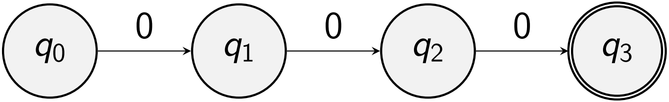

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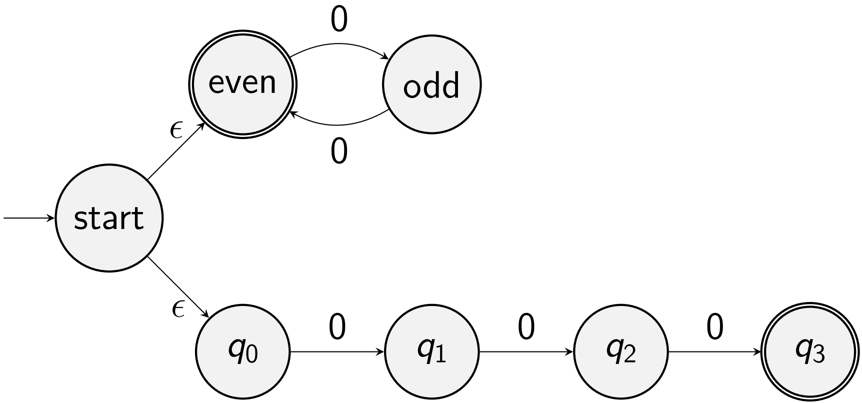

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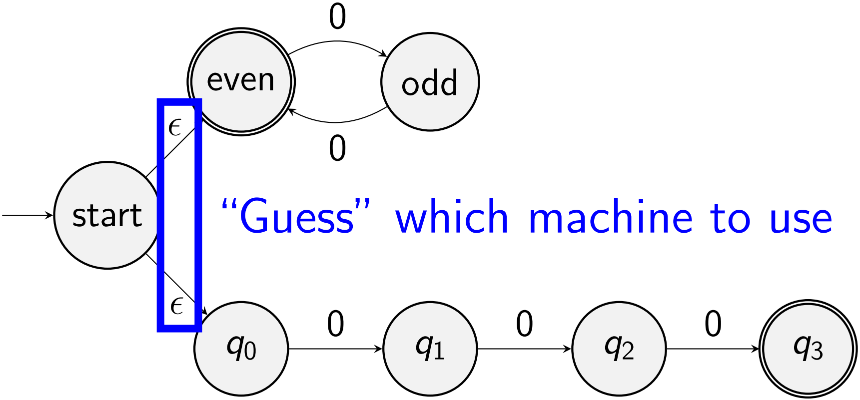

"Guess" which machine to use

Equivalence of NFAs and DFAs

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- Thus, there exists an NFA that recognizes L

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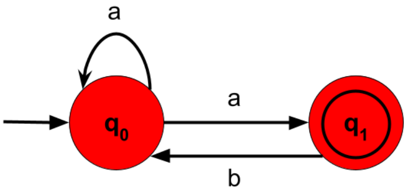
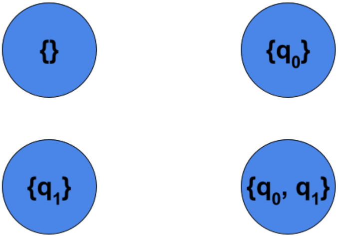
NFA to DFA conversion

Original NFA



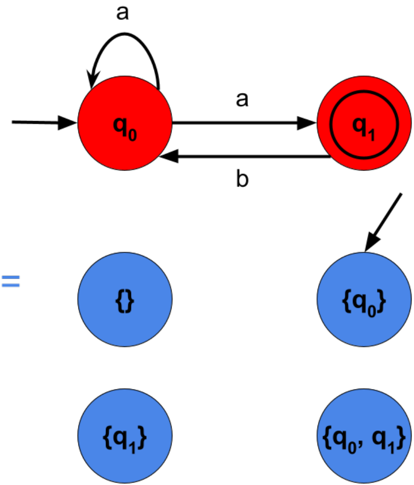
DFA States



NFA to DFA conversion

Original NFA
start state = q_0

DFA start state =
$\{q_0\}$



NFA to DFA conversion

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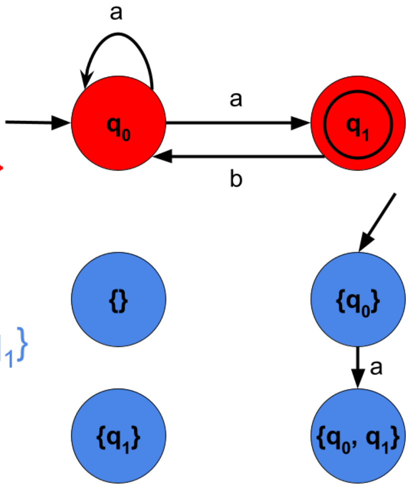
$\delta(q_0, a) = \{q_0, q_1\}$

DFA

$\delta(\{q_0\}, a) = \{q_0, q_1\}$



NFA to DFA conversion

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$\delta(q_0, b) = \{\}$

DFA

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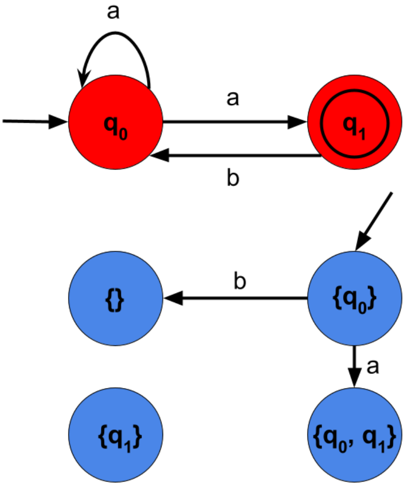

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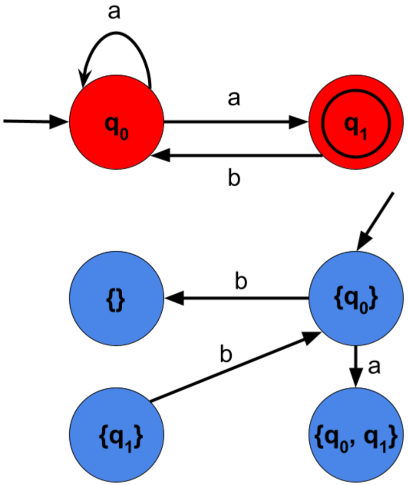


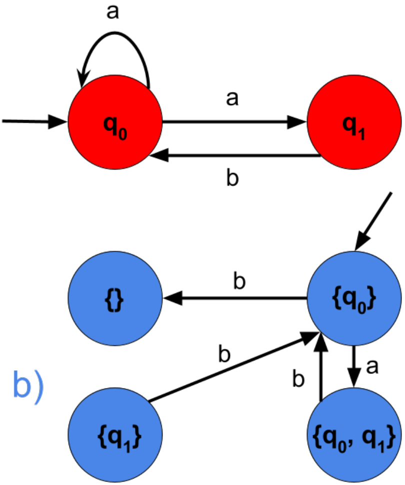
NFA to DFA conversion

Original NFA

$\delta(q_0, b) = \{\}$
$\delta(q_1, b) = \{q_0\}$

DFA

$\delta(\{q_0, q_1\}, b)$
$= \delta(q_0, b) \cup \delta(q_1, b)$
$= \{\} \cup \{q_0\}$
$= \{q_0\}$



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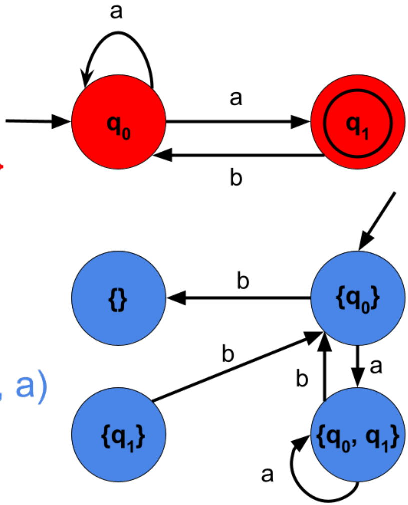

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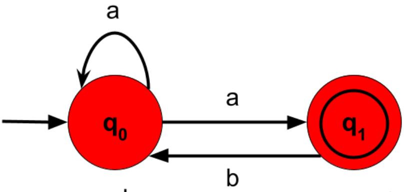
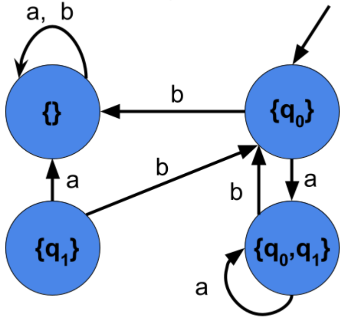
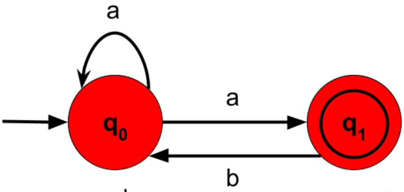


DFA with all transitions

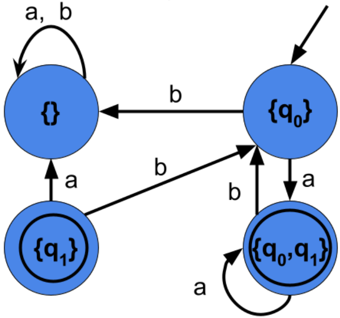


NFA to DFA conversion

Original NFA
F = $\{q_1\}$



DFA
F = anything containing an NFA accept state
= $\{\{q_1\}, \{q_0, q_1\}\}$



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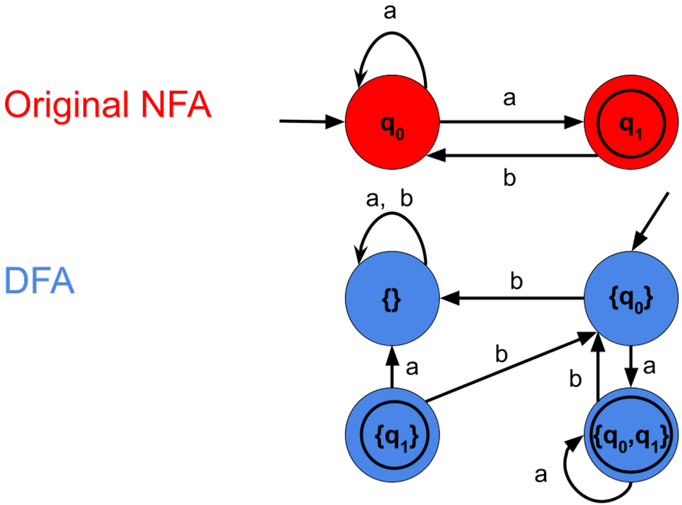

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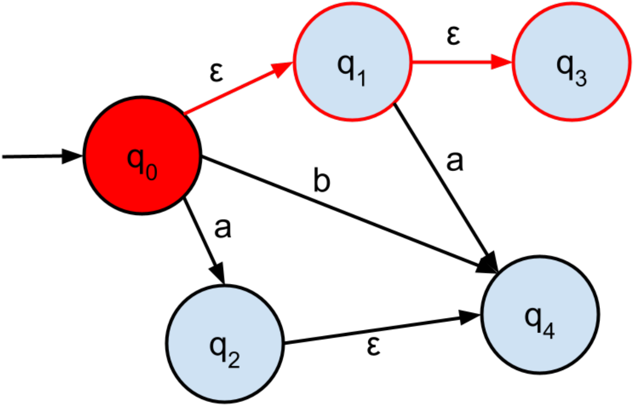
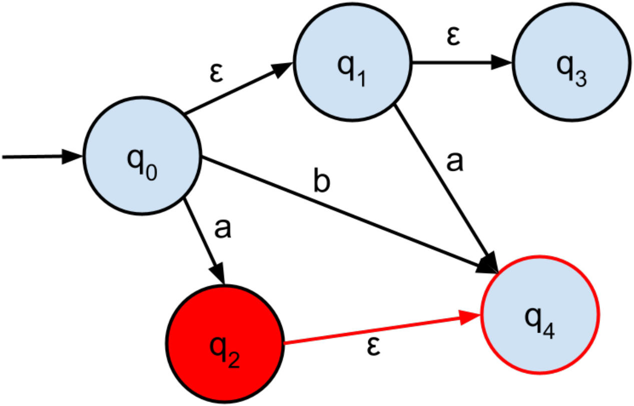
Epsilon Closure Example

$E(\{q_0\}) = \{q_0, q_1, q_3\}$



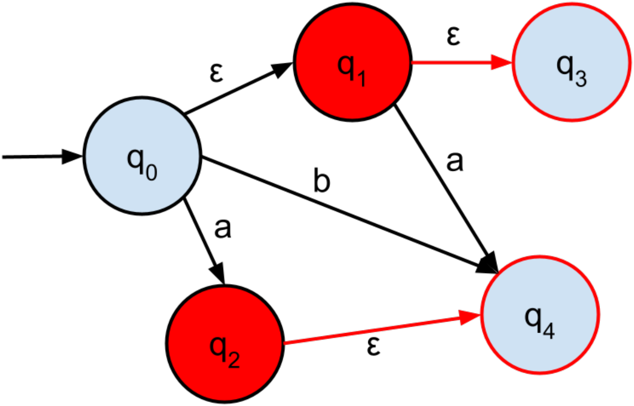
Epsilon Closure Example

$E(\{q_2\}) = \{q_2, q_4\}$



Epsilon Closure Example

$$E(\{q_1, q_2\}) = \{q_1, q_2, q_3, q_4\}$$



NFA to DFA conversion

How do we extend our conversion to account for ϵ transitions?

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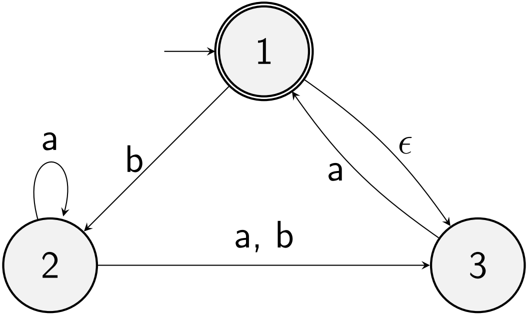
Let's convert the following NFA to a DFA



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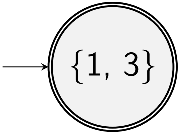

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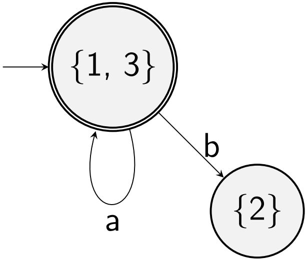

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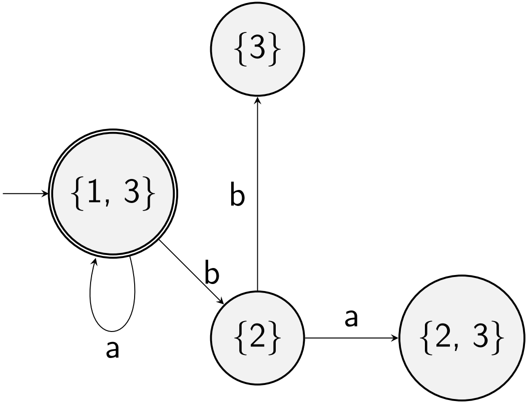

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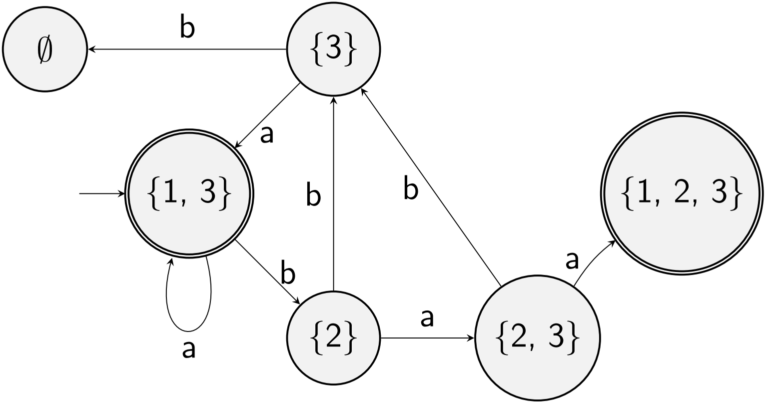

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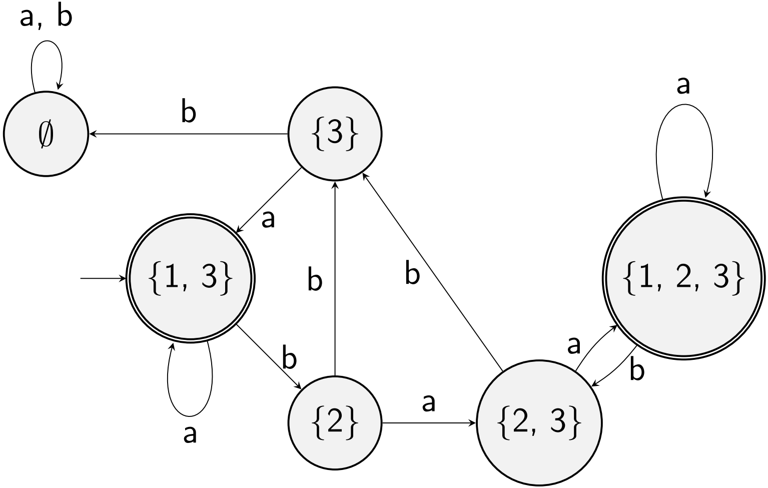

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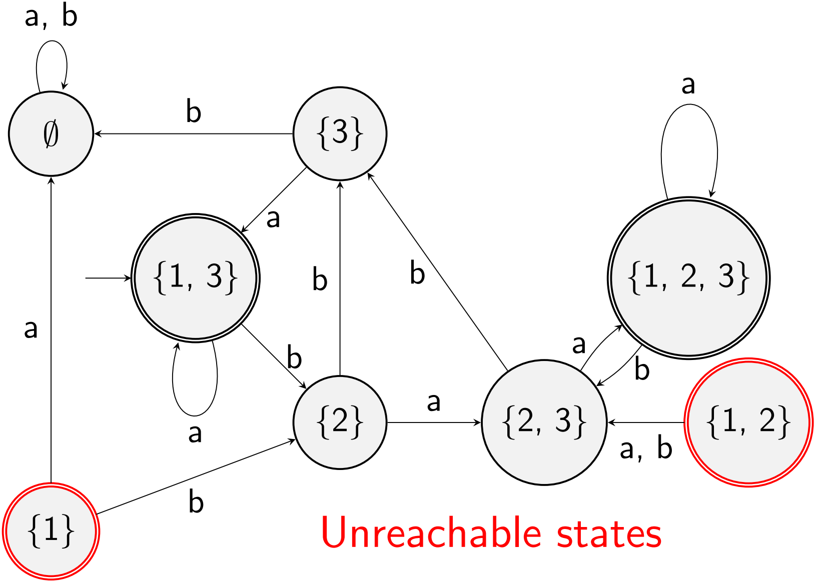

Unreachable states

NFAs and regular languages

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- **Corollary:** a language is regular if and only if it is recognized by an NFA
- It will often be more convenient use NFAs when we want to show that a langauge is regular!

Regular operations

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- **(Kleene) Star:**
  $A^* = \{\epsilon\} \cup \{w = w_1 w_2 \ldots w_n | w_i \in A\}$

Kleene's Theorem

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Theorem: The regular languages are closed under the regular operations

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  - For Kleene star we only modify the DFA for L_1

Closure under union

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Closure under union

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- Start with the two smaller NFAs

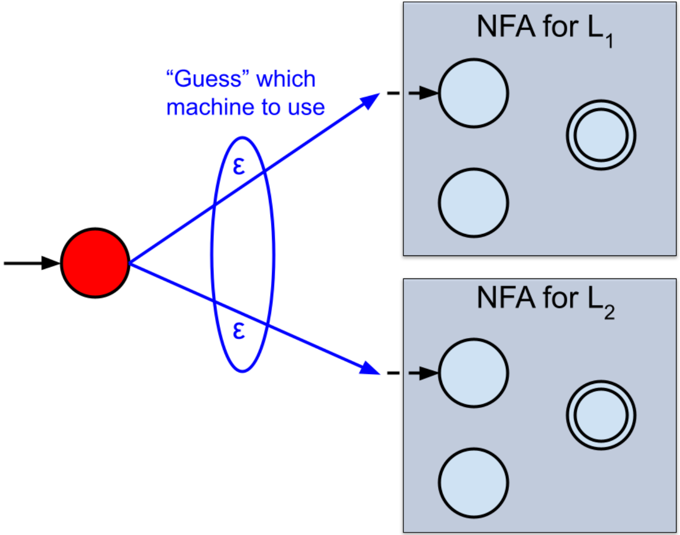
Closure under union

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add a new start state

Closure under union

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add a new start state
- Add ϵ transitions to the two original start states

Closure under union



Closure under concatenation

Closure under concatenation

- Let N_1 recognize L_1 and let N_2 recognize L_2

Closure under concatenation

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs

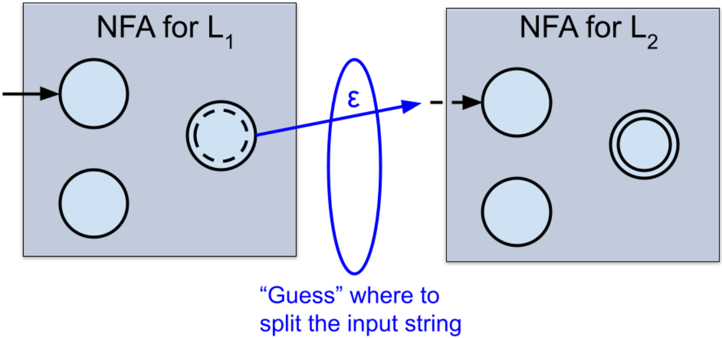
Closure under concatenation

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add an ϵ transition between N_1's accept state(s) and N_2's start state

Closure under concatenation

- Let N_1 recognize L_1 and let N_2 recognize L_2
- Start with the two smaller NFAs
- Add an ϵ transition between N_1's accept state(s) and N_2's start state
- Accept states in N_1 are no longer accept states (we have to accept in N_2)

Closure under concatenation



Closure under Kleene star

Closure under Kleene star

- Let N_1 recognize L_1

Closure under Kleene star

- Let N_1 recognize L_1
- Start with the smaller NFA

Closure under Kleene star

- Let N_1 recognize L_1
- Start with the smaller NFA
- Add ϵ transitions from each accept state back to the start state

Closure under Kleene star

- Let N_1 recognize L_1
- Start with the smaller NFA
- Add ϵ transitions from each accept state back to the start state
- Add an new start state with an ϵ transition to the original start state

Closure under Kleene star

- Let N_1 recognize L_1
- Start with the smaller NFA
- Add ϵ transitions from each accept state back to the start state
- Add an new start state with an ϵ transition to the original start state
  - This new start state will also be an accept state

Closure under Kleene star