Non-regular Languages

Arjun Chandrasekhar

Regular languages

Show that the following language is regular

$$\{0^n1^n | n \in \mathbb{N}\} = \{\epsilon, 01, 0011, 000111, \dots\}$$

- You may construct a DFA, NFA, or a regex to recognize the language

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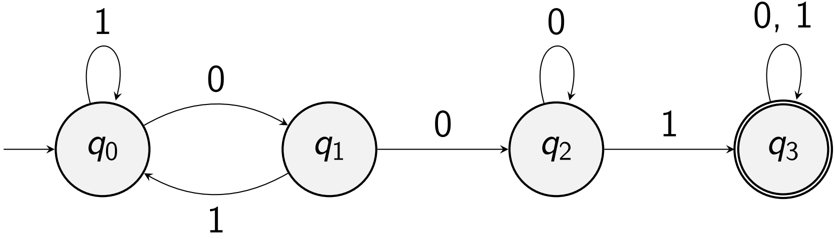
  Can DFAs be used to recognize *every* language?
- How do we show that a particular langauge cannot be recognized by *any* DFA?
  - We certainly can't exhaustively check every possible DFA...

Non-regul

- Are D … device?
  Can D … language?
- How c … ngauge
  canno
  - W … every
    p

Repeated states

How many symbols can we read before we are forced to repeat a state?



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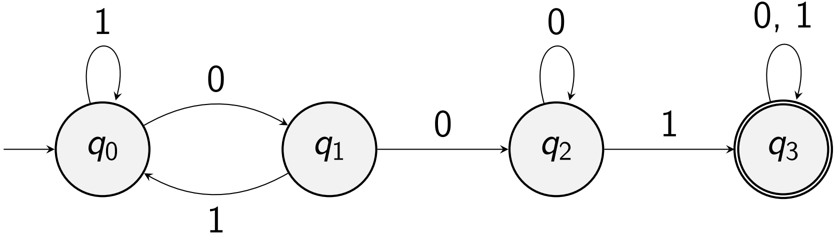

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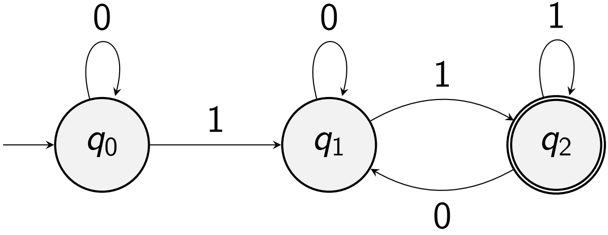

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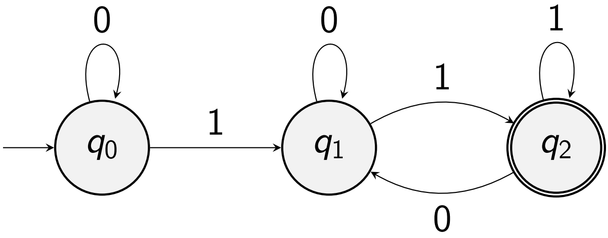

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  - At least one visited state is a repeat (pigeonhole principle)

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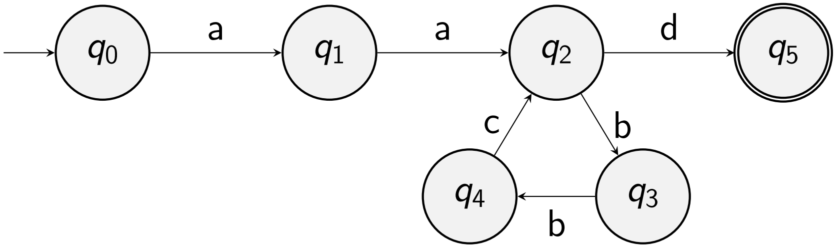

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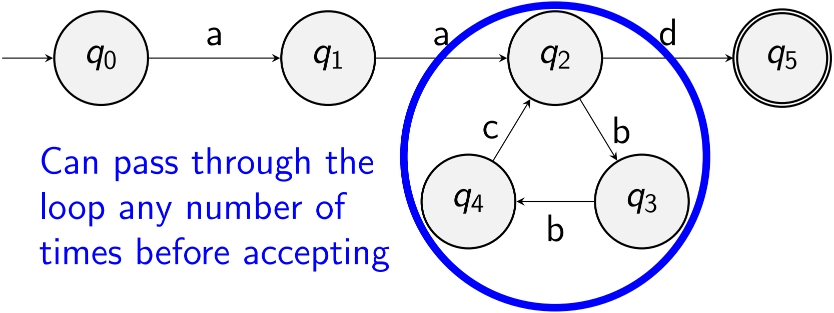

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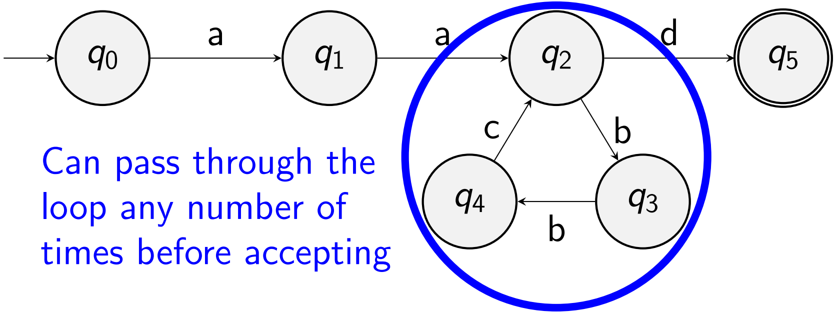

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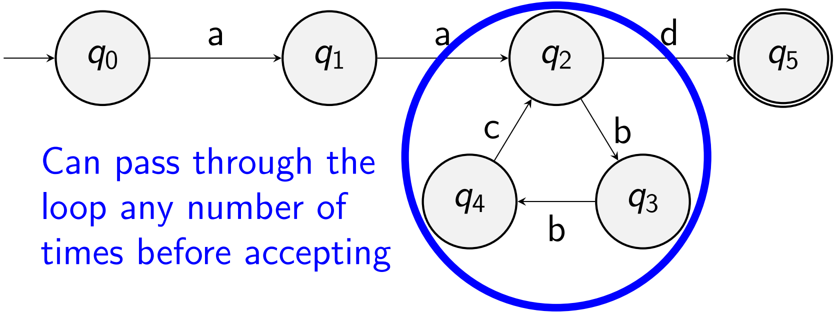

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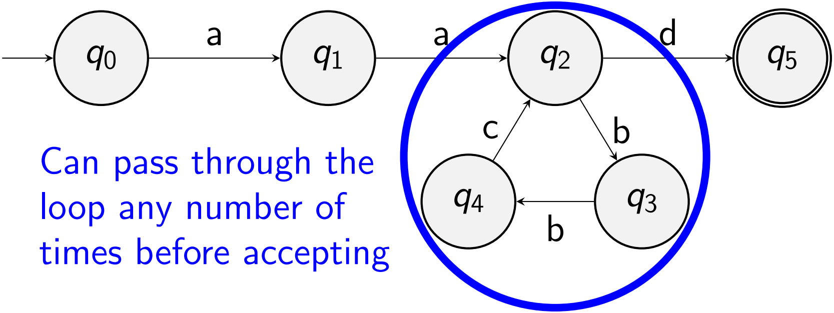

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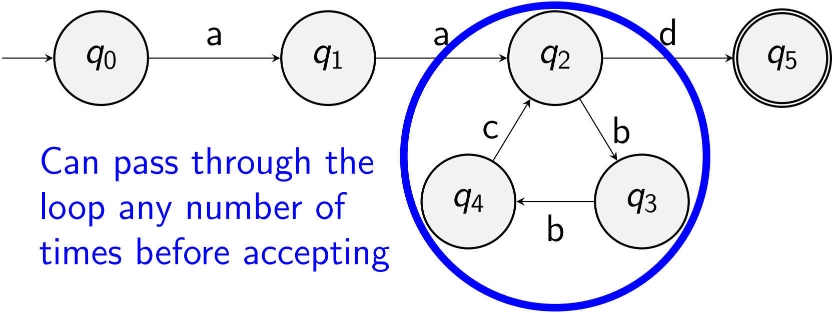

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- Let's consider the implications of one state being visited twice.

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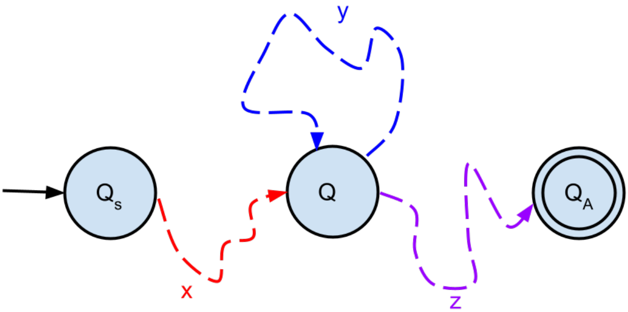

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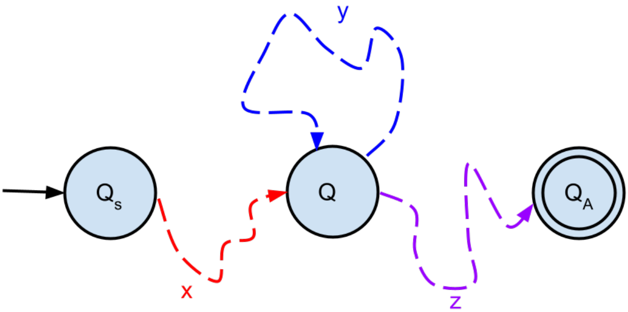

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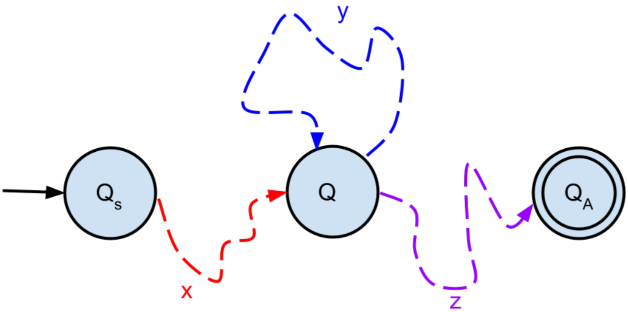

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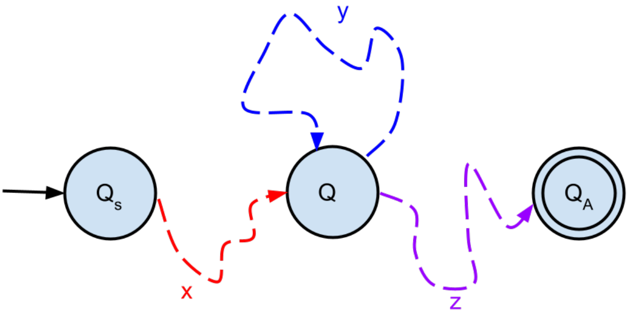

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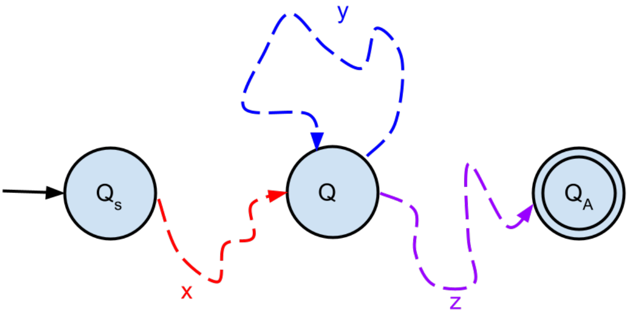

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  - We can add (or remove) any number of copies of y, and the new string will still be in the language

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- Thus, the "middle" (i.e. loop) part of the string can be pumped to make other strings that are accepted

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- Let $x = s_1 \ldots s_{j-1}, y = s_j \ldots s_{l-1}, z = s_l \ldots s_n$. Then $|y| > 0$, $|xy| \leq p$, and $xy^i z$ will be accepted for all i

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  - Thus the pumping lemma is satisfied.

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- **The contrapositive is that if L does not have a valid pumping length, it cannot possibly regular.**

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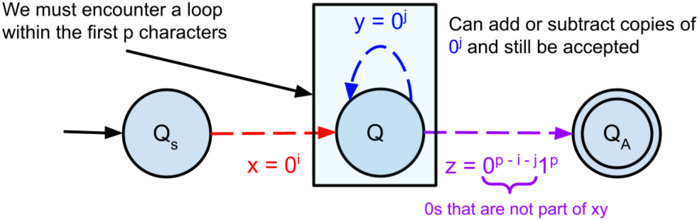
- If we pump y, the 0s and 1s won't be equal
- $w \in L$ is not pumpable, which contradicts the pumping lemma. We conclude L is not regular

The Pumping Lemma Example

$L = \{0^n 1^n \mid n \geq 0\}$

AFSOC L can be decided by a DFA with p states

$s = 0^p 1^p = \underbrace{0\ldots0}_{p}\underbrace{1\ldots1}_{p} \in L$



Any DFA that accepts $0^p 1^p$ would accept $0^i 0^j 0^j 0^{p-i-j} 1^p = 0^{p+j} 1^p \notin L$

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5. If Player 2 wins if s/he can pump to create a string $w' \notin L$. Otherwise, Player 1 wins

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  - There is always a valid way to split up *any* string of length $\geq p$ such that pumping won't cause a problem
- If L is not regular, Player 2 can win for *every* possible p
  - For any p, there exists a string $w \in L$ with length $\geq p$ that cannot be pumped no matter how it is split up

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4. No matter how Player 1 splits up w, Player 2 can pump up or down to make the 0s and 1s unequal

The pumping lemma

Proposition: Let $L = \{0^n 1^n | n \geq 0\}$. L is not a regular language.

1. Player 1 claims L is regular and declares a pumping length p
2. Player 2 picks $w = 0^p 1^p$
3. Player 1 splits up $w = xyz$ according to the pumping lemma rules
4. No matter how Player 1 splits up w, Player 2 can pump up or down to make the 0s and 1s unequal
5. Player 2 will win for every p, so we conclude L is not regular.

Pumping Lemma Example

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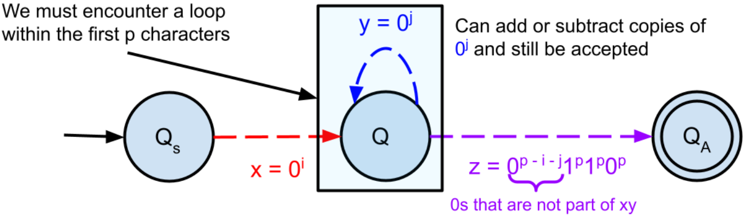
- If we pump y, the leading 0s won't equal the trailing 0s
- $w \in L$ is not pumpable; conclude L is not regular

Pumping Lemma Example

$L = \{ww^R \mid n \geq 0\}$

AFSOC L can be decided by a DFA with p states

$$s = 0^p1^p1^p0^p = \overbrace{\underbrace{0\ldots0}_{p}\underbrace{1\ldots1}_{p}}^{w}\overbrace{\underbrace{1\ldots1}_{p}\underbrace{0\ldots0}_{p}}^{w^R} \in L$$



Any DFA that accepts $0^p1^p1^p0^p$ would accept

$$0^i0^j0^j0^{p-i-j}1^p1^p0^p = 0^{p+j}1^p1^p0^p \notin L$$

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Let $\Sigma = \{a, b\}$. Let's prove that $L = \{a^i b^j | i \geq j\}$ is not regular

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Pumpi

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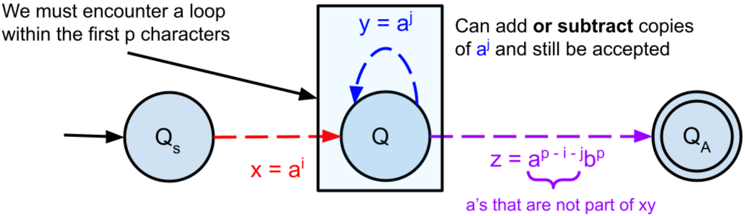

Any DFA that accepts $a^p b^p$ would accept

$$a^i \cancel{a^j} a^{p-i-j} b^p = a^i a^{p-i-j} b^p = a^{p-j} b^p \notin L$$

Pumping Lemma Example

Let $\Sigma = \{0, 1, +, =\}$. Consider the following language:

$\mathrm{ADD} = \{$i+j=k $|$ i, j, k are binary numbers, equation is valid$\}$

Which of the following strings are in the language?

A) $1100{+}11{=}1111$

B) $1001{+}0000{=}0110$

C) $1111{+}0{=}1111$

D) $001{=}111{+}111{=}000$

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Which of the following strings are in the language?

A) 1100+11=1111 ✓

B) 1001+0000=0110
Numbers don't add up

C) 1111+0=1111 ✓

D) 001=111+111=000
Wrong format

Pumping Lemma Example

Let's prove that ADD is not regular

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s: $1^p+0=1^p \in L$

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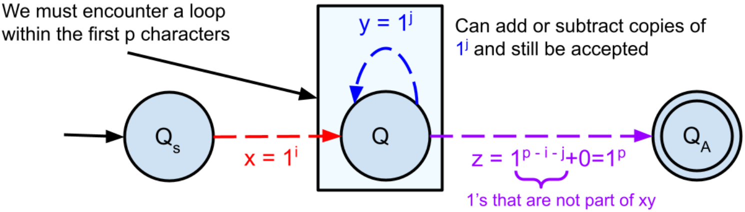

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$1^i1^j1^j1^{p-i-j}+0=1^p \rightarrow 1^{p+j}+0=1^p \notin L$