Theory of Computation
Time Complexity

Arjun Chandrasekhar

Introduction to complexity theory

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- **Complexity theory:** what problems can (and can't) be solved *within specific resource constraints*

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  - The input string may encode an object with a different size (e.g. graph with n vertices vs. adjacency matrix with n^2 elements)

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The machine runs in 5 seconds. Is that "fast"?

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None of these are properties of the actual *algorithm!*

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- The running time of an algorithm is the running time of a TM that implements the algorithm

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- What is the "order of magnitude" for the algorithm runtime?
- How does the algorithm "scale"?
  - As the input gets bigger, how many extra steps will the algorithm require?

Big-O Notation

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- We say $f(n)$ is $O(g(n))$ if there exists a constant c, and a cutoff point n_0, such that for all $n \geq n_0$
$$f(n) \leq c \cdot g(n)$$

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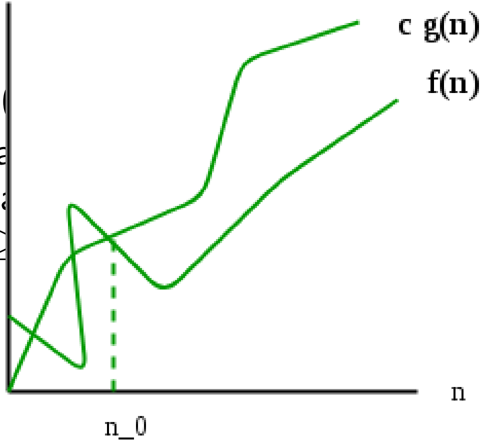

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- **Example:**

$$\begin{aligned} T(n) &= 5n^3 + 17n^2 \log(n) + 3.2n^{1.5} + 19747487584 \\ &\rightarrow 5n^3 \\ &\rightarrow O(n^3) \end{aligned}$$

Runtime Analysis Example

What is the time complexity of the following TM to decide $L = \{0^k 1^k | k \geq 0\}$?

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$O(n) + O(n) \cdot O(n) = O(n^2)$

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  - ...on a single-tape TM

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$O(n)$ to read the 0's
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- $O(2^{n^c})$ – "exponential"

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- For an algorithms course, we typically analyze complexity using models that are more expressive than a single-tape TM

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Theorem: Any language that can be recognized by a k-tape TM in $O(T(n))$ time can be recognized by a single-tape TM in $O(T(n)^2)$ time

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- **Remark:** If a TM runs in $O(T(n))$ time, it touches at most $O(T(n))$ tape squares

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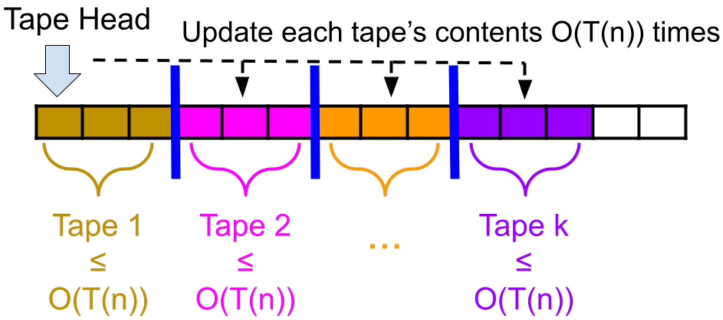

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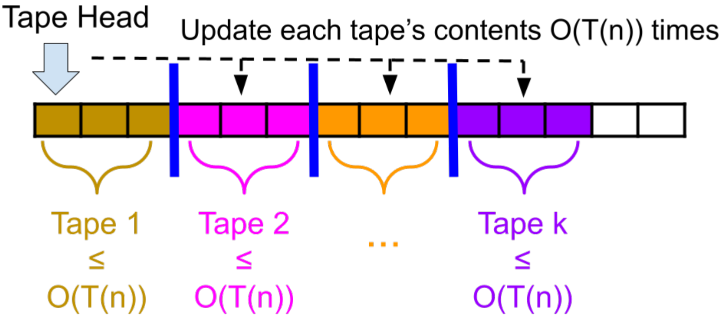

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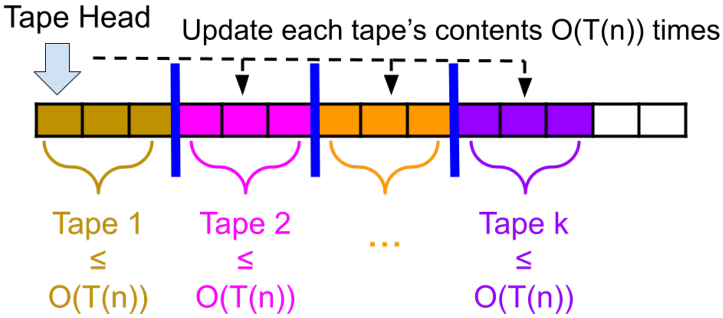

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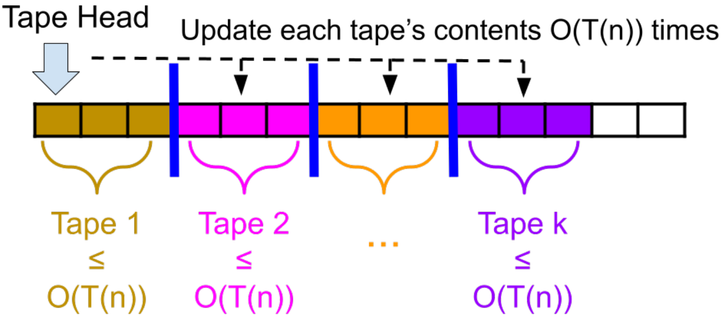

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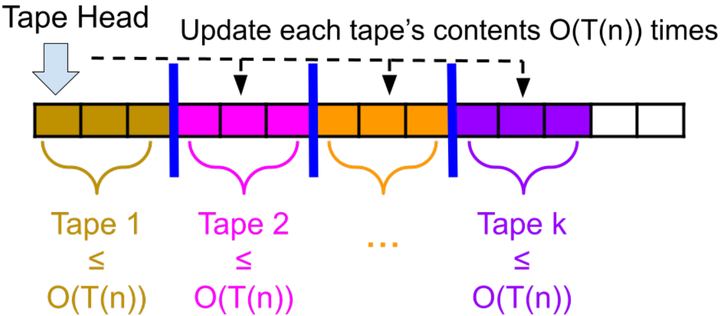

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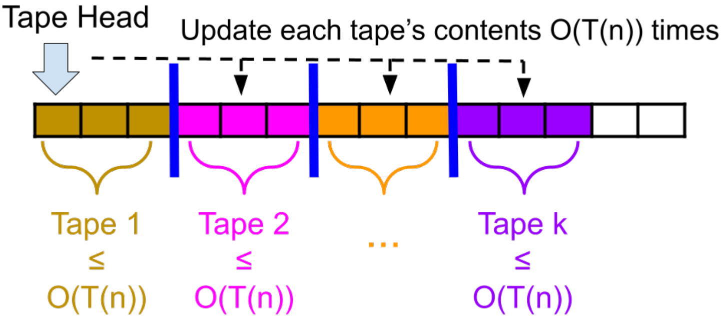

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- We will see that this is good enough for the problems we are exploring in this course

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- **TMs formalize our intuitive notion of (efficient) algorithms**
- Quantum computers may prove to be an exception